J. Math. Study
doi: 10.4208/jms.v53n3.20.02

Vol. **53**, No. 3, pp. 247-264
September 2020

Multigrid Method for a Two Dimensional Fully Nonlinear Black-Scholes Equation with a Nonlinear Volatility Function

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Received May 31, 2019; Accepted October 22, 2019;
Published online May 14, 2020.

Abstract. This paper deals with the task of pricing European basket options in the presence of transaction costs. We develop a model that incorporates the illiquidity of the market into the classical two-assets Black-Scholes framework. We perform a numerical simulation using finite difference method. We consider a nonlinear multigrid method in order to reduce computational costs. The objective of this paper is to investigate a deterministic extension for the Barles' and Soner's model and to demonstrate the effectiveness of multigrid approach to solving a fully nonlinear two dimensional Black-Scholes problem.

AMS subject classifications: 65N55, 65N06, 35K55, 65BXX

Key words: Fully nonlinear equation, multigrid method, black-scholes equation, finite difference method, FAS algorithm.

1 Introduction

Basket option is a financial derivative where the underlying asset is in the form of a sum or an average of different assets gathered together in a basket. Pricing basket options for nonlinear Black-Scholes equations is an interesting problem in financial mathematics, since an analytical solution of the basket option does not exist even in the linear case. Therefore, finding numerical solutions becomes an essential approach for solving the governing pricing equations (see [2]). In this work we are concerned with pricing European options in incomplete markets, more precisely, markets that are subjected to transaction costs. In this case the Black-Scholes theory given in [7] fails. To overcome this problem, several researchers have developped some extensions to the Black-Scholes equation

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by incorporating more realistic assumptions on financial markets (see [10, 11, 13, 20] for more details). We focus our attention on a nonlinear deterministic Black-Scholes extension given by Barles and Soner in [6]. They proposed an enlarged volatility to capture the influence of transaction costs,

$$V_t+\frac{1}{2}\tilde{\sigma}\left(S,t,\frac{\partial^2 V}{\partial S^2}\right)^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0, \qquad S>0, \quad t\in(0,T), \tag{1.1}$$

where

$$\tilde{\sigma}\left(S,t,\frac{\partial^2 V}{\partial S^2}\right)^2=\sigma^2\left(1+\Psi(e^{r(T-t)}a^2S^2\frac{\partial^2 V}{\partial S^2})\right). \tag{1.2}$$

The unknown V is the European call option, S is the underlying asset, T is the expiration date of the option V, σ is the constant volatility given in the linear Black-Scholes model, $r\geq 0$ is the risk-free interest rate constant, a is a nonnegative parameter that summarizes risk aversion and transaction costs and Ψ denotes the solution to the nonlinear ordinary differential equation,

$$\begin{cases} \Psi'(x)=\dfrac{\Psi(x)+1}{2\sqrt{x\Psi(x)}-x}, \qquad x\neq 0, \\ \\ \Psi(0)=0. \end{cases} \tag{1.3}$$

Figure 1 is a representation of the solution of ODE (1.3) using a fourth order Runge-Kutta method. The problem (1.1) has been studied theoretically in [6] and the existence of a unique continuous viscosity solution has been shown using the theory of stochastic optimal control [14]. Recently, a constructive theoretical approach for the problem (1.1) was given in [1]. For numerical simulations of option pricing problems, Monte-Carlo methods are widely used. They are easy to implement and can handle multiple dimensions but they suffer from slow convergence (see [15,16]). Over the last two decades, numerical methods based on PDEs have also gained popularity since they allow an easy computation of Greeks which represent a basic instrument for risk management. However, PDE methods cannot compete with Monte-Carlo methods in two or more dimensions in terms of computing time, consequently, they are limited to one or two dimensions. The aim of our work is to push forward the limit of what can be done using PDE methods. There is a considerable literature that reviews numerical simulations of the option price when markets are subject to transaction costs using PDEs, we quote [4, 19, 22], where several nonlinear Black-Scholes models in the presence of transaction costs have been studied for European and American options, [30] presented a positivity preserving scheme for the Barles' and Soner's model and they studied its convergence. [21] used a grid stretching technique on non-uniform meshes for the computation and [24, 25] reviewed nonlinear parabolic equations arising from markets under transaction costs. However, there are only few works for the computation of nonlinear Black-Scholes equations depending on more than one underlying asset, among them we cite [29], they gave a numerical