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A priori Error Analysis of a Discontinuous Galerkin Method for Cahn–Hilliard–Navier–Stokes Equations

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Abstract. In this paper, we analyze an interior penalty discontinuous Galerkin method for solving the coupled Cahn–Hilliard and Navier–Stokes equations. We prove unconditional unique solvability of the discrete system, and we derive stability bounds without any restrictions on the chemical energy density function. The numerical solutions satisfy a discrete energy dissipation law and mass conservation laws. Convergence of the method is obtained by obtaining optimal a priori error estimates.

AMS subject classifications: 35G25, 65M60, 65M12, 76D05

Key words: Cahn–Hilliard–Navier–Stokes, interior penalty discontinuous Galerkin method, existence, uniqueness, stability, error estimates.

1 Introduction

The Cahn–Hilliard–Navier–Stokes system strikes an optimal balance in terms of thermodynamical rigor and computational efficiency for modeling immiscible two-phase flow. The model that belongs to the class of diffuse interface or phase-field methods, has been used in physics, chemistry, biology, and engineering fields. In recent years, driven by the major developments of numerical algorithms and by increased availability of computational resources, direct numerical simulation of Cahn–Hilliard–Navier–Stokes equations has become increasingly popular [1, 14, 16, 25, 27, 30].

This paper is devoted to the numerical analysis of an interior penalty discontinuous Galerkin method for the coupled Cahn–Hilliard and Navier–Stokes equations in two and three dimensional domains. The class of discontinuous Galerkin methods belongs to the class of locally mass conservative numerical methods. In addition, local mesh refinement with hanging nodes and high order approximation are easily handled by these methods. The unknowns are approximated by discontinuous piecewise polynomials. For the

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Cahn–Hilliard–Navier–Stokes system, the unknowns are the order parameter, the chemical potential, the fluid velocity and pressure. We prove existence and uniqueness of the discrete solution and we show stability of the scheme for any chemical energy density function. At the continuous level, the physical quantities satisfy an energy dissipation law, and at the discrete level, we show the energy decreases with time. Similarly, the solutions satisfy global and local mass conservation because of the use of discontinuous approximation spaces. A priori error estimates show convergence of the numerical method with optimal rates.

The convergence analysis of continuous finite element method for the Cahn–Hilliard–Navier–Stokes model has been extensively investigated. In the work of Feng [12], continuous $\mathbb{P}_2-\mathbb{P}_0$ elements are used for the approximation of the velocity and pressure whereas continuous $\mathbb{P}_r$ elements, for $r\geq 1$ are used for the approximation of the chemical potential and order parameter. Convergence of the solution is obtained via a compactness argument. Kay, Styles, and Welford in [24] analyze semi-discrete and fully discrete finite element schemes in two-dimensional computational domains. Under a CFL-like condition, they obtain a priori error estimates for the semi-discrete method and a convergence proof based on a compactness argument for the fully discrete scheme. Diegel, Wang, Wang, and Wise in [9] analyze a second order in time mixed finite element method, based on Crank–Nicolson method. Continuous $\mathbb{P}_r$ elements are used for the chemical potential, order parameter and pressure whereas continuous $\mathbb{P}_{r+1}$ are used for the velocity with any positive integer r. The work contains unconditional energy stability and optimal error estimates. In [5, 22], a projection method is used to handle the Navier–Stokes equations. Han and Wang introduce a second order in time method and show unconditional unique solvability of the algorithm. The work [22] does not contain any theoretical proof of convergence of the solution. Cai and Shen obtain unconditional unique solvability, derive error estimates and show a convergence analysis based on a compactness argument. In [5], both chemical potential and order parameter are approximated by continuous $\mathbb{P}_2$ elements and the velocity and pressure are approximated by a stable pair of finite element spaces. In addition of using continuous finite elements in space, all the works mentioned above assume a special form of chemical energy density, namely a double-well potential, also called Ginzburg–Landau potential. The coupling term in the momentum equation of the Navier–Stokes system may take several forms, that are equivalent at the continuous level but that yield different numerical schemes at the discrete level. We note that in [5, 9, 12], the coupling term is the product of the chemical potential and the gradient of the order parameter. In the other works [22, 24] as well as in our present work, the coupling term is the product of the order parameter and the gradient of the chemical potential.

To the best of our knowledge, this work is the first theoretical analysis for a fully discrete interior penalty discontinuous Galerkin scheme of the Cahn–Hilliard–Navier–Stokes system. However, the literature on numerical methods for solving the Cahn–Hilliard equation (resp. the Navier–Stokes equations) is abundant. Finite element methods and interior penalty discontinuous Galerkin methods have been employed for each