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Finding the Maximal Eigenpair for a Large, Dense, Symmetric Matrix based on Mufa Chen's Algorithm

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Abstract. A hybrid method is presented for determining maximal eigenvalue and its eigenvector (called eigenpair) of a large, dense, symmetric matrix. Many problems require finding only a small part of the eigenpairs, and some require only the maximal one. In a series of papers, efficient algorithms have been developed by Mufa Chen for computing the maximal eigenpairs of tridiagonal matrices with positive off-diagonal elements. The key idea is to explicitly construct effective initial guess of the maximal eigenpair and then to employ a self-closed iterative algorithm. In this paper, we will extend Mufa Chen's algorithm to find maximal eigenpair for a large scale, dense, symmetric matrix. Our strategy is to first convert the underlying matrix into the tridiagonal form by using similarity transformations. We then handle the cases that prevent us from applying Chen's algorithm directly, e.g., the cases with zero or negative super- or sub-diagonal elements. Serval numerical experiments are carried out to demonstrate the efficiency of the proposed hybrid method.

AMS subject classifications: 60J60, 34L15

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1 Introduction

The best approach for computing all the eigenpairs (eigenvalues and eigenvectors) of a dense symmetric matrix involves three steps:

- *reduction*: reduce the given symmetric matrix A to tridiagonal form T (i.e. nonzero elements only occur on the diagonal and super-/sub-diagonals);
- *solution of tridiagonal eigen-problem*: compute all the eigenpairs of T;
- *back-transformation*: map the eigenvectors of T into those of A.

For an $N\times N$ matrix, the reduction and back-transformation steps require $\mathcal{O}(N^3)$ arithmetic operations each. Note that most algorithms for the tridiagonal eigen-problem also had cubic complexity in the worst case, including the QR algorithm and inverse iteration. As pointed out by [18] the tridiagonal problem can be the computational bottleneck for large problems taking nearly 70$\sim$80% of the total time to solve the entire dense problem. As a result, numerous methods exist for the numerical computation of the eigenvalues of a real tridiagonal matrix to high accuracy, see, e.g., [2,10,11]. To find eigenvalues of a symmetric tridiagonal matrix typically requires $\mathcal{O}(N^2)$ operations [8], although fast algorithms exist which require $\mathcal{O}(N\ln N)$ [6].

In practice, many problems require finding only a small part of the eigenvalues and eigenvectors; for such problems, finding all the eigenpairs may be time consuming and wasteful. The problem of computing the maximal eigenpairs has been a classical subject, and the methods for this problem that are discussed most are the power method, the inverse method, the Rayleigh quotient method, and some hybrid method, see, e.g., [1,12,13]. Finding the largest eigenpairs has many applications in signal processing, control, and recent development of Google's PageRank algorithm. In a series of papers, Mufa Chen [3–5] developed some efficient algorithms for computing the maximal eigenpairs for tridiagonal matrices. The key idea is to explicitly construct effective initials for the maximal eigenpairs and also employ a self-closed iterative algorithm. The algorithm makes the total number of iterations independent of the size of the matrix.

In [14], the authors extended Chen's algorithm to deal with large scale tridiagonal matrices. By using appropriate scalings and by optimizing numerical complexity, we make the computational cost for each iteration to be $\mathcal{O}(N)$. As a result, it requires $\mathcal{O}(N)$ number of computational cost to obtain the maximal eigenpair.

The main purpose of this paper is to compute the maximal eigenpair for large, dense, symmetric matrices by further extending Chen's algorithm. Namely, for the three steps mentioned above, we first convert the given symmetric matrix to a