INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 17, Number 2, Pages 151–171

AN ERROR ESTIMATE OF A EULERIAN-LAGRANGIAN LOCALIZED ADJOINT METHOD FOR A SPACE-FRACTIONAL ADVECTION DIFFUSION EQUATION

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Abstract. We derive a Eulerian-Lagrangian localized adjoint method (ELLAM) for a space-fractional advection diffusion equation that includes a fractional Laplacian operator for modeling such application as a superdiffusive advective transport. The method symmetrizes the numerical scheme and generates accurate numerical solutions even if large time steps and relatively coarse grid meshes are used. We also study the structure of the stiffness matrix to further reduce the computational complexity and memory requirement. We prove an error estimate for the ELLAM. Numerical experiments are presented to show the potential of the method.

Key words. Space-fractional advection diffusion, fractional Laplacian, characteristic method, error estimate, superdiffusive transport.

1. Introduction

Advection diffusion partial differential equations (PDEs) model advective diffusive transport in porous media, stochastic dynamics and other applications [3, 10, 12]. The traditional integer-order advection diffusion PDEs, which can be viewed as the Fokker-Planck PDEs of the Ito stochastic processes driven by Brownian motion, were shown to provide accurate description of Fickian diffusive transport in relatively homogeneous porous media. However, in strongly heterogeneous porous media, the underlying particle motions exhibit superdiffusive transport behavior that has an algebraic decaying heavy tail and so has a large deviation from the Brownian motion. Consequently, space-fractional advection diffusion PDEs were shown to provide an accurate description of the superdiffusive transport [13].

It is well known, even in the context of the traditional integer-order advection diffusion PDEs, conventional numerical methods tend to generate some combination of nonphysical oscillations and excessive numerical diffusion [5, 19]. Eulerian-Lagrangian methods provide a competitive means for accurately and efficiently solving these problems [2, 9, 8]. These methods exhibit the advantages of alleviating the Courant number restrictions and reducing the time truncation errors. Namely, they can produce accurate numerical solutions even if the mesh is coarse and the time step is large. There are two principal drawbacks of the Eulerian-Lagrangian method, i.e., it is failure to conserve mass and it is difficult to treat various boundary conditions. However, for advection-dominated problems, the ELLAM can overcome the two principal shortcomings of Eulerian-Lagrangian method, while maintaining their advantages [11]. In this paper we derive a ELLAM for a space-fractional advection-diffusion PDE and prove its error estimate. In the framework of the ELLAM [5], the advective component is treated by a characteristic tracking algorithm and the diffusive component is treated separately by using a more standard spatial approximation, i.e., the Eulerian-Lagrangian methods combine the convection and capacity terms in the governing equation to carry out the temporal discretization in

Received by the editors August 15, 2018.
2000 *Mathematics Subject Classification.* 35R20, 26A33, 65M12.

a Lagrangian coordinate, and discretize the standard or anomalous diffusion term on a fixed mesh. In other words, the characteristic methods change an fractional advection diffusion equation into a fractional diffusion equation, which transports along with the characteristic curves. We also analyze the structure of its stiffness matrix to develop a fast solution method for the resulting linear algebraic system with a full stiffness matrix. Finally, we conduct some numerical examples to verify the accuracy of the ELLAM scheme and the efficiency of the fast solution method.

The remainder of this paper is organized as follows. We begin in section 2 by giving the nonlocal model and some preliminaries. In section 3, we derive the ELLAM scheme for the fractional equation. We provide an error estimate for the ELLAM scheme in section 4. Section 5 investigates the structure of the coefficient matrix and section 6 proves an auxiliary lemma used in section 4. In section 7, we conduct some numerical tests. Finally, we summarize some remarks.

2. Model Problem and Preliminaries

We consider the following space-fractional advection diffusion transport PDE

$$
\begin{aligned}
&p_t + (V(x,t)p)_x - d\,p_{xx} + \gamma(-\Delta)^{\frac{\alpha}{2}} p = f(x,t),\ x \in \mathbb{R},\ t \in (0,T], \\
&p(x,t) = 0, \quad x \notin (a,b),\ t \in (0,T], \quad p(x,0) = p_0(x), \quad x \in \mathbb{R},
\end{aligned} \tag{1}
$$

where

$$
(-\Delta)^{\frac{\alpha}{2}} p(x,t) = C_\alpha \int_{\mathbb{R}} \frac{p(x,t) - p(y,t)}{|x-y|^{1+\alpha}} dy,\ \alpha \in (0,2), \tag{2}
$$

with $C_\alpha = \frac{\alpha}{2^{1-\alpha}\sqrt{\pi}} \frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}$. In such application as advective diffusive transport, $p(x,t)$ usually represents the concentration of the solute or solvent in the fluid, $V(x,t)$ refers to the velocity field of the fluid, $-dp_{xx}$ models the Fickian diffusive transport, $\gamma(-\Delta)^{\frac{\alpha}{2}} p(x,t)$ models the superdiffusive transport, and $f(x,t)$ represents the source term. Here d and γ are nonnegative constants. In stochastic dynamics, $p(x,t)$ is the probability density function that describes the ensemble of realizations of a Lévy process, $-dp_{xx}$ models the Brownian motion component and $V(x,t)$ is the drift. $p_0(x) \geq 0$ is the initial configuration of the model which satisfies the constraint

$$
\int_{\mathbb{R}} p_0(x) dx = 1.
$$

Since $p(x,t)$ is zero outside the interval (a,b) for any time $t \in (0,T]$, we just consider this model on the interval (a,b) in this paper.

2.1. Sobolev Spaces and Approximation Properties. First, let $W_p^k(a,b)$ consist of functions whose weak derivatives up to order-k are p-th Lebesgue integrable in (a,b). Let $H^k(a,b) := W_2^k(a,b)$

$$
\|v\|_{H^k(a,b)} := \Big(\|v\|^2_{H^{k-1}(a,b)} + \Big\| \frac{d^k v}{dx^k} \Big\|^2_{L^2(a,b)} \Big)^{1/2}.
$$

For any Banach space X, we introduce Sobolev spaces involving time

$$
W_p^k(t_1,t_2;X) := \Big\{ f : \Big\| \frac{\partial^\beta f}{\partial t^\beta}(\cdot,t) \Big\|_X \in L^p(t_1,t_2),\ 0 \leq \beta \leq k,\ 1 \leq p \leq \infty \Big\},
$$

$$
\|f\|_{W_p^k(t_1,t_2;X)} := \begin{cases} \Big(\displaystyle\sum_{\beta=0}^k \int_{t_1}^{t_2} \Big\| \frac{\partial^\beta f}{\partial t^\beta}(\cdot,t) \Big\|_X^p dt \Big)^{1/p}, & 1 \leq p < \infty, \\ \displaystyle\max_{0 \leq \beta \leq k} \ \text{ess} \sup_{(t_1,t_2)} \Big\| \frac{\partial^\beta f}{\partial t^\beta}(\cdot,t) \Big\|_X, & p = \infty. \end{cases}
$$