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The Existence of Weak Solutions of a Higher Order Nonlinear Eilliptic Equation

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Abstract: In this paper, we show the existence of weak solutions for a higher order nonlinear elliptic equation. Our main method is to show that the evolution operator satisfies the fixed point theorem for Banach semilattice.

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1 Introduction and Main Result

In this paper, we consider the following elliptic problem:

$$(-\Delta)^m u = \lambda V(x)u + h(u), \quad x \in \mathbf{R}^N,\ N \geq 3, \tag{P_λ}$$

where λ is a positive parameter, $m \in \mathbf{N}$, $0 < m < \dfrac{N}{2}$. We make the assumptions over V and h as:

(V) $V(x) \in C(\mathbf{R}^N, \mathbf{R}^+)$, and there exists an $a \in \mathbf{R}$ such that

$$\lim\sup_{|x|\to\infty} V(x)|x|^a < \infty, \tag{1.1}$$

the index a describes the property of V near infinity, and we assume $a > 2m$.

(H) $h : \mathbf{R}^N \to \mathbf{R}$ is a measurable function, and the function $h(s)$ is nondecreasing on $\mathbf{R}$, there exists a positive constant C such that h holds the growth condition:

$$|h(s)| \leq C|s|^{\frac{N+2m}{N-2m}}, \quad s \in \mathbf{R}. \tag{1.2}$$

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Let $C_c^\infty(\mathbf{R}^N)$ denote the collection of smooth functions with compact support. Denote by $\mathcal{D}^{m,2}(\mathbf{R}^N)$ the completion of $C_c^\infty(\mathbf{R}^N)$ under the norm

$$\|u\| = \begin{cases} \left(\int_{\mathbf{R}^N} |\Delta^k u|^2 \mathrm{d}x\right)^{\frac{1}{2}} & \text{if } m = 2k; \\ \left(\int_{\mathbf{R}^N} |\nabla(\Delta^k u)|^2 \mathrm{d}x\right)^{\frac{1}{2}} & \text{if } m = 2k+1. \end{cases}$$

It is easy to see that the above norm comes from the scalar product

$$(u,\, v)_{\mathcal{D}^{m,\,2}(\mathbf{R}^N)} = \begin{cases} \int_{\mathbf{R}^N} (\Delta^k u)(\Delta^k v) \mathrm{d}x & \text{if } \ m = 2k; \\ \int_{\mathbf{R}^N} \nabla(\Delta^k u)\cdot\nabla(\Delta^k v) \mathrm{d}x & \text{if } \ m = 2k+1. \end{cases}$$

In this work, we consider our model problem in the working space $\mathcal{D}^{m,\,2}(\mathbf{R}^N)$.

The main result of the paper is the following theorem:

Theorem 1.1 *If the assumptions (V) and (H) hold, then there exists a positive constant $\lambda_* > 0$ such that for any $0 < \lambda < \lambda_*$ the model problem (P_λ) has a nontrivial weak solution.*

There are many methods to study nonlinear elliptic equations, such as the theory of monotone operators, the Schauder's fixed point theorem for compact mapping, the upper and lower solutions method, non-smooth critical point theories etc., we can see [1] for the choice of the method.

In this paper, we use the method introduced by Heikkilä[2] to show the existence of weak solutions of the elliptic problem, the main work is to show that the developed operator is increasing in order Banach spaces, then, due to the fixed point theorem, we get that the solution of the model problem is just the fixed point. The approach has been used in many work, one can see [3]–[9]. Here we give a solution in higher-order Sobolev space.

For the paper, the letter C is a positive constant which maybe variety at different lines. We denote the norm of the Lebesgue space $L^p(\mathbf{R}^N)$ $(1 < p < \infty)$ as

$$\|\cdot\|_p = \left(\int_{\mathbf{R}^N} |\cdot|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

2 Preliminary

In this section, we give some preliminary results which are used later. Firstly, we give a special case of the result on Sobolev embedding which one can see [6].

Lemma 2.1 *The space $\mathcal{D}^{m,\,2}(\mathbf{R}^N)$ is continuously embedding into the Lebesgue space $L^{\frac{2N}{N-2m}}(\mathbf{R}^N)$.*

Next, we give the definition of weak solution of the problem (P_λ) in $\mathcal{D}^{m,2}(\mathbf{R}^N)$.

Definition 2.1 *A function $u \in \mathcal{D}^{m,\,2}(\mathbf{R}^N)$ is a weak solution of the problem (P_λ) if there holds*

$$\int_{\mathbf{R}^N} (-\Delta)^m uv \mathrm{d}x = \lambda \int_{\mathbf{R}^N} V(x)uv \mathrm{d}x + \int_{\mathbf{R}^N} h(u)u \mathrm{d}x.$$