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Location of Zeros for the Weak Solution to a p-Ginzburg-Landau Problem

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Abstract: This paper is concerned with the asymptotic behavior of the solution u_ε of a p-Ginzburg-Landau system with the radial initial-boundary data. The author proves that the zeros of u_ε in the parabolic domain $B_1(0) \times (0, T]$ locate near the axial line $\{0\} \times (0, T]$. In particular, all the zeros converge to this axial line when the parameter ε goes to zero.

Key words: p-Ginzburg-Landau equation, initial-boundary value problem, location of zero

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1 Introduction

Let $n \geq 3$ and $B = \{x \in \mathbf{R}^n;\ |x| < 1\}$. Write $B_T = B \times (0, T]$, where $T \in (0, \infty)$. We are concerned with the asymptotic behavior of the weak solution u_ε of the following problem:

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \frac{1}{\varepsilon^p}u(1 - |u|^2), \qquad (x, t) \in B_T, \tag{1.1}$$

$$u(x, t)_{\partial B} = x, \qquad t \geq 0, \tag{1.2}$$

$$u(x, 0) = \frac{x}{|x|}, \qquad x \in B, \tag{1.3}$$

when $\varepsilon \to 0$. Recall that a weak solution of (1.1) is a measurable function $u : B_T \to \mathbf{R}^n$, such that

$$\operatorname*{esssup}_{t\in(0,T)} \|u(\,\cdot\,, t)\|^2_{L^2(B)} + \|\nabla u\|^p_{L^p(B_T)} < \infty,$$

and for any $\phi \in C_0^\infty(B_T)$,

$$\int_{B_T} u\phi_t \mathrm{d}x\mathrm{d}t = \int_{B_T} |\nabla u|^{p-2}\nabla u \nabla\phi \mathrm{d}x\mathrm{d}t - \frac{1}{\varepsilon^p}\int_{B_T} u\phi(1 - |u|^2)\mathrm{d}x\mathrm{d}t. \tag{1.4}$$

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Moreover, if a function u is a weak solution of (1.1), and $\lim\limits_{t\to 0^+}\int_B |u(x,\, t)-\frac{x}{|x|}|\mathrm{d}x=0$, then u solves (1.1)–(1.3) in weak sense.

In the case of $p=n=2$, the problem can be used to described the properties of vortices in the study of the phase transition, such as the theories of superconductor, superfluids and XY-magnetism (see [1] and the references therein).

When $\max\left\{2,\, \frac{n}{2}\right\} < p < n$, the author proves that the zeros of u_ε in the parabolic domain $B_1(0)\times(0,\, T]$ locate near the axial line $\{0\}\times(0,\, T]$. In addition, the author also consider the Hölder convergence of the solution when the parameter ε tends to zero (see [2]). Indeed, the restriction $\max\left\{2,\, \frac{n}{2}\right\} < p < n$ is not essential. It comes from the technique deficiency in the proof of Proposition 2.3 in [2]. We expect those results in [2] are still true in the wider extent for the value of p.

We shall prove the following theorems:

Theorem 1.1 *Assume* $1<p\le n$, *and let* u_ε *be the weak solution of* (1.1)–(1.3). *Then for any* $\sigma>0$, *there exists a constant* h (*independent of* ε) *such that*

$$\left\{(x,\, t)\in B\times[0,\, T];\, |u_\varepsilon(x,\, t)|<\frac{1}{2}\right\}\subset \Omega,$$

where

$$\Omega=(\overline{B(0,\, h\varepsilon)}\times[\sigma,\, T])\cup(\overline{B(0,\, \sigma)}\times[0,\, \sigma]),\qquad p\in(2,\, n);$$

$$\Omega=(\overline{B(0,\, h\varepsilon)}\times[0,\, T])\cup(B\times[0,\, \sigma]),\qquad p=n \text{ or } p\in(1,\, 2).$$

Remark 1.1 Theorem 1.1 implies that all the zeros of u_ε are located near $\{0\}\times[0,\, T]$ when $\varepsilon\to 0$ and σ is sufficiently small. Namely, there does not exist any zero in the domain far away from $\{0\}\times[0,\, T]$ since

$$|u_\varepsilon(x,\, t)|\ge\frac{1}{2},\qquad (x,\, t)\in\Omega. \tag{1.5}$$

Theorem 1.2 *Under the same assumption of Theorem* 1.1, *for any* $\sigma>0$, *there exists a* $C>0$ *such that*

$$\sup_{t\in[\sigma,T]}\left[\int_0^t\int_0^1\left|\frac{\partial}{\partial\tau}u_\varepsilon(x,\tau)\right|^2\mathrm{d}x\mathrm{d}\tau+E_\varepsilon(u_\varepsilon(x,t),B)\right]\le C,\qquad p\in(1,n); \tag{1.6}$$

$$\sup_{t\in[\sigma,T]}\left[\int_0^t\int_0^1\left|\frac{\partial}{\partial\tau}u_\varepsilon(x,\tau)\right|^2\mathrm{d}x\mathrm{d}\tau+E_\varepsilon(u_\varepsilon(x,t),B\setminus B(0,\sigma))\right]\le C,\qquad p=n, \tag{1.7}$$

where

$$E_\varepsilon(u,B)=\frac{1}{p}\int_B|\nabla u|^p\mathrm{d}x+\frac{1}{4\varepsilon^p}\int_B(1-|u|^2)^2\mathrm{d}x.$$

By the same argument in [2], from (1.5)–(1.7) we can also derive the Hölder convergence of ∇u_ε when $2<p\le n$. Moreover, we also have the following convergence result:

Theorem 1.3 *Assume that* $\max\left\{1,\, \frac{8}{n+2}\right\}<p\le n$, *and* u_ε *is the weak solution of* (1.1)–(1.3). *We have*

$$\lim_{\varepsilon\to 0}\nabla u_\varepsilon=\nabla\frac{x}{|x|}$$

in $C^{\alpha,\frac{\alpha}{2}}_{\mathrm{loc}}([\overline{B}\setminus\{0\}]\times(0,\, T],\, \mathbf{R}^n)$ *for some* $\alpha\in(0,\, 1)$.