COMMUNICATIONS
IN
MATHEMATICAL
RESEARCH
34(2)(2018), 106–116

On the Group of p-endotrivial kG-modules

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Communicated by Du Xian-kun

Abstract: In this paper, we define a group $T_p(G)$ of p-endotrivial kG-modules and a generalized Dade group $D_p(G)$ for a finite group G. We prove that $T_p(G) \cong T_p(H)$ whenever the subgroup H contains a normalizer of a Sylow p-subgroup of G, in this case, $K(G) \cong K(H)$. We also prove that the group $D_p(G)$ can be embedded into $T_p(G)$ as a subgroup.

Key words: p-endotrivial module, the group of p-endotrivial modules, endo-permutation module, Dade group

2010 MR subject classification: 20C05, 20C20

Document code: A

Article ID: 1674-5647(2018)02-0106-11

DOI: 10.13447/j.1674-5647.2018.02.02

1 Introduction

The (absolutely) p-divisible kG-module is introduced by Benson and Carlson[1]. It is a tool to study the decomposition of the tensor product of two kG-modules, and a tool to study nilpotent elements in the Green ring. Different from many other kinds of kG-modules, its definition is independent of many classical aspects for the group algebra kG, but only essentially depended on the prime number p, and its class is a big one, all (relative) projective kG-modules are p-divisible.

The endotrivial kG-module is named by Dade[2]. It is a building block for the endo-permutation modules which are the sources for the irreducible modules of many finite groups (see [3]), and it forms an important part for the Picard group of self-equivalences of the stable module category. In this paper, based on the p-divisible kG-module, we extend the ordinary endotrivial kG-module and the relative endotrivial kG-module to the p-endotrivial kG-module (see [2]–[4]).

The (indecomposable) p-endotrivial kG-module here, at the same time, is a special kind of the splitting trace module (see [5]), that is, the kG-module V such that the trace map

Received date: Nov. 21, 2016.
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$Tr\colon \mathrm{End}(V) \to k$ is split, and Auslander and Carlson[5] proved that the tensor of the splitting trace module V with the almost-split sequence for the trivial kG-module k cannot be split.

Here we focus on the group $T_p(G)$ of p-endotrivial kG-modules; on the one hand, we study the restriction map for this group, and prove that if the subgroup H contains the normalizer of a Sylow p-subgroup of G, for example, H is a strongly embedded subgroup of G, then $T_p(G) \cong T_p(H)$ (Theorem 2.8), and $K(G) \cong K(H)$ (Theorem 2.9); on the other hand, by using $T_p(G)$ we obtain a generalized Dade group $D_p(G)$ for the finite group G, and prove that $D_p(G)$ can be embedded into $T_p(G)$ as a subgroup (Theorem 3.3). Our results extend the results for the group $T(G)$ of endo-trivial kG-modules and the results for the Dade group $D(P)$ for the finite p-group P.

Throughout the paper, we fix a prime number p, a finite group G such that $p \,|\, |G|$, and an algebraic closed field k of characteristic p. All modules are finitely generated, and p divides the order of any finite group involve in a p-endotrivial kG-module. For the necessary terminologies in the paper the reader can consult [6].

2 The Group $T_p(G)$ of p-endotrivial kG-modules

For a prime number p and a finite group G with $p \,|\, |G|$, we say that a kG-module V is a p-divisible kG-module if the dimension of any indecomposable direct summand of V is divisible by p.

The terminology of the p-divisible kG-module is introduced to be an absolutely p-divisible kG-module (see [1]); confined to the algebraic closed field k, any indecomposable kG-module is already absolutely indecomposable therein, so the p-divisible kG-module is also the absolutely p-divisible kG-module therein.

Remark 2.1 The class of p-divisible kG-modules is a big one, any (relative) projective kG-module is p-divisible (see [7], Exer. 23.1), but the trivial kG-module k is not p-divisible. The direct summand of a p-divisible kG-module, the direct sum of two p-divisible kG-modules, the tensor product of a p-divisible kG-module and a kG-module, remain to be p-divisible (see [1], Proposition 2.2). Sometimes we denote a p-divisible kG-module with p-divisible for short.

Definition 2.1 *Let V be a kG-module. If the endomorphism module of V can be regarded as the direct sum of the trivial module k and a p-divisible kG-module, that is,*

$$\mathrm{End}_k(V) \cong k \oplus_k U,$$

where U is a p-divisible kG-module, then we say that V is a p-endotrivial kG-module, or V is p-endotrivial.

Remark 2.2 (1) Here, for the endomorphism module $\mathrm{End}_k(V)$, $g \cdot f := gfg^{-1}$, $g \in G$, $f \in \mathrm{End}_k(V)$, and for any (indecomposable) p-endotrivial kG-module V, $k \,|\, \mathrm{End}_{kG}(V)$, the trace map ($Tr\colon \mathrm{End}_k(V) \to k$) is a split surjection, so the tensor product of V with the almost-split sequence for the trivial kG-module k must fail to split (see [5]).