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Unique Common Fixed Points for Two Weakly C^*-contractive Mappings on Partially Ordered 2-metric Spaces

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Abstract: In this paper, we give existence theorems of common fixed points for two mappings with a weakly C^*-contractive condition on partially ordered 2-metric spaces and give a sufficient condition under which there exists a unique common fixed point.

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1 Introduction and Preliminaries

Gähler[1]–[3] introduced the definition of 2-metric spaces and discussed the existence problems of fixed points. From then on, many authors discussed and obtained the existence problems of coincidence points and (common) fixed points with a variety of different forms. Especially, there have appeared a lot of useful results in recent years, see the references [4]–[16] and the related papers. All these results generalize and improve the corresponding fixed point theorem in metric spaces.

Definition 1.1 [1]–[3] *A 2-metric space (X, d) consists of a nonempty set X and a function $d: X \times X \times X \to [0, +\infty)$ such that*

(i) *for distant elements $x, y \in X$, there exists a $u \in X$ such that $d(x, y, u) \neq 0$;*
(ii) *$d(x, y, z) = 0$ if and only if at least two elements in $\{x, y, z\}$ are equal;*
(iii) *$d(x, y, z) = d(u, v, w)$, where $\{u, v, w\}$ is any permutation of $\{x, y, z\}$;*
(iv) *$d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$ for all $x, y, z, u \in X$.*

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Definition 1.2 [1]–[3] *A sequence $\{x_n\}_{n\in\mathbf{N}_+}$ in 2-metric space $(X,\, d)$ is said to be a Cauchy sequence if for each $\varepsilon > 0$ there exists a positive integer $N \in \mathbf{N}_+$ such that $d(x_n,\, x_m,\, a) < \varepsilon$ for all $a \in X$ and $n, m > N$. A sequence $\{x_n\}_{n\in\mathbf{N}_+}$ is said to be convergent to $x \in X$ if for each $a \in X$, $\lim\limits_{n\to+\infty} d(x_n,\, x,\, a) = 0$. And we write that $x_n \to x$ and call x the limit of $\{x_n\}_{n\in\mathbf{N}_+}$. A 2-metric space $(X,\, d)$ is said to be complete if every Cauchy sequence in X is convergent.*

Choudhury[17] introduced the next definition in a real metric space:

Definition 1.3 [17] *Let $(X,\, d)$ be a metric space and $T\colon X \to X$ be a map. T is said to be weak C-contraction if there exists a continuous function $\varphi\colon [0,\, +\infty)^2 \to [0,\, +\infty)$ with $\varphi(s,\, t) = 0 \Longleftrightarrow s = t = 0$ such that*

$$d(Tx,\, Ty) \le \frac{1}{2}[d(x,\, Ty) + d(y,\, Tx)] - \varphi(d(x,\, Ty),\ d(y,\, Tx)), \qquad x, y \in X.$$

Choudhury[17] also proved that any map satisfying the weak C-contraction has a unique fixed point on a complete metric space (see [17], Theorem 2.1). Later, the above result was extended to the case in a complete ordered metric spaces (see [18], Theorems 2.1, 2.3 and 3.1).

In 2013, Definition 1.3 was extended to the case in a 2-metric space by Dung and Hang[10] as follows:

Definition 1.4 [10] *Let $(X,\, \preceq,\, d)$ be a ordered 2-metric space, $T\colon X \to X$ a map. T is said to be weak C-contraction if there exists a continuous function $\varphi\colon [0,\, +\infty)^2 \to [0,\, +\infty)$ with $\varphi(s,\, t) = 0 \Longleftrightarrow s = t = 0$ such that for any $x, y, a \in X$ with $x \preceq y$ or $y \preceq x$,*

$$d(Tx,\, Ty,\, a) \le \frac{1}{2}[d(x,\, Ty,\, a) + d(y,\, Tx,\, a)] - \varphi(d(x,\, Ty,\, a),\ d(y,\, Tx,\, a)).$$

Dung and Hang[10] proved that any weakly C-contractive map has fixed points on complete ordered 2-metric spaces (see [10], Theorems 2.3, 2.4 and 2.5). The results generalized and improved the corresponding conclusions in [17]–[18].

Definition 1.5 *Let $(X,\, \preceq,\, d)$ be a ordered 2-metric space and $S, T\colon X \to X$ be two maps. S, T are said to be weakly C^*-contractive maps if there exists a continuous function $\varphi\colon [0,\, +\infty)^2 \to [0,\, +\infty)$ with $\varphi(s,\, t) = 0 \Longleftrightarrow s = t = 0$ such that for any $x, y, a \in X$ with $x \preceq y$ or $y \preceq x$,*

$$d(Sx,\, Ty,\, a) \le kd(x,\, y,\, a) + l[d(x,\, Ty,\, a) + d(y,\, Sx,\, a)] - \varphi(d(x,\, Ty,\, a),\ d(y,\, Sx,\, a)),$$

where k and l are two real numbers satisfying $l > 0$ and $0 < k + l \le 1 - l$.

Obviously, if $S = T$ and $k = 0$ and $l = \dfrac{1}{2}$, then Definition 1.5 becomes Definition 1.3.

Definition 1.6 [10] *Let $(X,\, d)$ be a 2-metric space and $a, b \in X$, $r > 0$. The set*

$$B(a,\, b;\, r) = \{x \in X : d(a,\, b,\, x) < r\}$$

is said to be a 2-ball with centers a and b and radius r. Each 2-metric d on X generalizes a topology τ on X whose base is the family of 2-balls. τ is said to be a 2-metric topology.