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On Lie 2-bialgebras

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Abstract: A Lie 2-bialgebra is a Lie 2-algebra equipped with a compatible Lie 2-coalgebra structure. In this paper, we give another equivalent description for Lie 2-bialgebras by using the structure maps and compatibility conditions. We can use this method to check whether a 2-term direct sum of vector spaces is a Lie 2-bialgebra easily.

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1 Introduction

1.1 Background

This paper is a sequel to [1], in which the notion of Lie 2-bialgeras was introduced. The main purpose of this paper is to give an equivalent condition for Lie 2-bialgebras. Generally speaking, a Lie 2-bialgebra is a Lie 2-algebra endowed with a Lie 2-coalgebra structure, satisfying certain compatibility conditions. As we all know, a Lie bialgebra structure on a Lie algebra $(\mathfrak{g}, [\,\cdot\,, \cdot\,])$ consists of a cobracket $\delta\colon \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$, which squares to zero, and satisfies the compatibility condition: for all $x, y, z \in \mathfrak{g}$,

$$\delta([x,\, y]) = [x,\, \delta(y)] - [y,\, \delta(x)].$$

Consequently, one may ask what is a Lie 2-bialgebra. A Lie 2-bialgebra is a pair of 2-terms of L_∞-algebra structure underlying a 2-vector space and its dual. The compatibility conditions are described by big bracket (see [1]). And an L_∞-algebra structure on a $\mathbb{Z}$-graded vector space can be found in [2]–[4]. This description of Lie 2-bialgebras seems to be elegant, but one cannot get directly the maps twisted between them and compatibility conditions. This is what we will explore in this paper.

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This paper is organized as follows: In Section 1, we recall the notion of big bracket, which has a fundamental role in this paper. Then, we introduce the basic concepts in Section 2 which is closely related to our result, that is, Lie 2-algebras and Lie 2-coalgebras, most of which can be found in [3]. Finally, in Section 3, we give an equivalent description of Lie 2-bialgebras, whose compatibility conditions are given by big bracket.

1.2 The Big Bracket

We introduce the following Notations.

(1) Let V be a graded vector space. The degree of a homogeneous vector $\boldsymbol{e}$ is denoted by $|\boldsymbol{e}|$.

(2) On the symmetric algebra $\mathscr{S}^\bullet(V)$, the symmetric product is denoted by $\odot$.

It is now necessary to recall the notion of big bracket underlying the graded vector spaces [1]. Let $V=\bigoplus_{k\in\mathbf{Z}} V_k$ be a $\mathbf{Z}$-graded vector space, and $V[i]$ be its degree-shifted one. Now, we focus on the symmetric algebra $\mathscr{S}^\bullet(V[2]\oplus V^*[1])$, denoted by $\mathscr{S}^\bullet$. In order to equip $\mathscr{S}^\bullet$ with a Lie bracket, i.e., the Schouten bracket, denoted by $\{\,\cdot\,,\,\cdot\,\}$, we define a bilinear map $\{\,\cdot\,,\,\cdot\,\}\colon \mathscr{S}^\bullet\otimes\mathscr{S}^\bullet\to\mathscr{S}^\bullet$ by:

(1) $\{v,\, v'\}=\{\varepsilon,\,\varepsilon'\}=0$, $\{v,\,\varepsilon\}=(-1)^{|v|}\langle v\mid\varepsilon\rangle$, $v, v'\in V[2]$, $\varepsilon,\,\varepsilon'\in V^*[1]$;

(2) $\{e_1,\,e_2\}=-(-1)^{(|e_1|+3)(|e_2|+3)}\{e_2,\,e_1\}$, $e_i\in\mathscr{S}^\bullet$;

(3) $\{e_1,\,e_2\odot e_3\}=\{e_1,\,e_2\}\odot e_3+(-1)^{(|e_1|+3)|e_2|}e_2\odot\{e_1,\,e_3\}$, $e_i\in\mathscr{S}^\bullet$.

Clearly, $\{\,\cdot\,,\,\cdot\,\}$ has degree 3, and all homogeneous elements $e_i\in\mathscr{S}^\bullet$ satisfy the following modified Jacobi identity:

$$\{e_1,\,\{e_2,\,e_3\}\}=\{\{e_1,\,e_2\},\,e_3\}+(-1)^{(|e_1|+3)(|e_2|+3)}\{e_2,\,\{e_1,\,e_3\}\}. \tag{1.1}$$

Hence, $(\mathscr{S}^\bullet,\,\odot,\,\{\,\cdot\,,\,\cdot\,\})$ becomes a Schouten algebra, or a Gerstenhaber algebra, see [1] and [4] for more details. Note that the big bracket here is different from that in [5], which is defined on $\mathscr{S}^\bullet(V\oplus V^*)$ without degree shifting.

For element $F\in S^p(V[2])\odot S^q(V^*[1])$, we define the following multilinear map: for all $x_i\in\mathscr{S}^\bullet(V[2])$,

$$D_F\colon\ \underbrace{\mathscr{S}^\bullet(V[2])\otimes\cdots\otimes\mathscr{S}^\bullet(V[2])}_{q\text{-tuples}}\to\mathscr{S}^\bullet(V[2])$$

by

$$D_F(x_1,\cdots,x_q)=\{\{\cdots\{\{F,\,x_1\},\,x_2\},\cdots,x_{q-1}\},\,x_q\}.$$

Lemma 1.1 *The following equations hold*:

(1) $|D_F(x_1,\,x_2,\,\cdots,\,x_q)|=|x_1|+|x_2|+\cdots+|x_q|+|F|+3q$;

(2) $D_F(x_1,\,\cdots,\,x_i,\,x_{i+1},\,\cdots,\,x_q)=(-1)^{(|x_i|+3)(|x_{i+1}|+3)}D_F(x_1,\,\cdots,\,x_{i+1},\,x_i,\,\cdots,\,x_q)$.

Proof. Since the degree of big bracket is 3, we apply this fact q-times to obtain (1).

If $q=2$, by (1.1), we have

$$\begin{aligned}\{x_1,\,\{x_2,\,F\}\}&=\{\{x_1,\,x_2\},\,F\}+(-1)^{(|x_1|+3)(|x_2|+3)}\{x_2,\,\{x_1,\,F\}\}\\&=(-1)^{(|x_1|+3)(|x_2|+3)}\{x_2,\,\{x_1,\,F\}\}.\end{aligned}$$