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# One Parameter Deformation of Symmetric Toda Lattice Hierarchy

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**Abstract:** In this paper, we study one parameter deformation of full symmetric Toda hierarchy. This deformation is induced by Hom-Lie algebras, or is the applications of Hom-Lie algebras. We mainly consider three kinds of deformation, and give solutions to deformations respectively under some conditions.
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## 1 Introduction

Consider the following equation given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L} = \boldsymbol{BL} - \boldsymbol{LB} = [\boldsymbol{B},\ \boldsymbol{L}], \tag{1.1}$$

where $\boldsymbol{L}$ is an $n \times n$ symmetric real tridiagonal matrix, and $\boldsymbol{B}$ is the skew symmetric matrix obtained from $L$ by

$$\boldsymbol{B} = \boldsymbol{L}_{>0} - \boldsymbol{L}_{<0},$$

where $\boldsymbol{L}_{>0(<0)}$ denotes the strictly upper (lower) triangular part of $\boldsymbol{L}$. In order to study the Toda lattice of statistical mechanics, the equation (1.1) was introduced by Flaschka[1], and this further was studied by Kodama *et al.*[2],[3].

The notion of Hom-Lie algebras was introduced by Hartwig *et al.*[4] as part of a study of deformations of the Witt and the Virasoro algebras. In a Hom-Lie algebra, the Jacobi identity is twisted by a linear map, called the Hom-Jacobi identity. Some $q$-deformations of the Witt and the Virasoro algebras have the structure of a Hom-Lie algebra (see [4] and [5]). Because of close relation to discrete and deformed vector fields and differential calculus

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(see [4], [6] and [7]), more people pay special attention to this algebraic structure and their representations (see [8] and [9]).

We give an application of Hom-Lie algebras. Define a smooth map

$$\beta : R_1 \longrightarrow GL(V), \qquad \boldsymbol{\beta}(s) \in GL(V),$$

where $R_1$ is a subset of $\mathbf{R}$, $s \in R_1$. In this paper, $R_1 = \mathbf{R} \setminus \{0\}$, or $R_1 = \mathbf{R}$.

We mainly consider the following system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L} = \boldsymbol{\beta}(s)\boldsymbol{B}\boldsymbol{\beta}(s)^{-1}\boldsymbol{L}\boldsymbol{\beta}(s)^{-1} - \boldsymbol{\beta}(s)\boldsymbol{L}\boldsymbol{\beta}(s)^{-1}\boldsymbol{B}\boldsymbol{\beta}(s)^{-1} = [\boldsymbol{B},\ \boldsymbol{L}]_{\boldsymbol{\beta}(s)}, \tag{1.2}$$

where $s \in R_1$, and $s$ is not dependent on variable $t$. $[\ \cdot\ ,\ \cdot\ ]_{\boldsymbol{\beta}(s)}$ is just a Hom-Lie bracket, and $(\mathfrak{gl}(V),\ [\ \cdot\ ,\ \cdot\ ]_{\boldsymbol{\beta}(s)},\ \mathrm{Ad}_{\boldsymbol{\beta}(s)})$ is a Hom-Lie algebra (see [9]), where $\mathrm{Ad}_{\boldsymbol{\beta}(s)}(\boldsymbol{L}) = \boldsymbol{\beta}(s)\boldsymbol{L}\boldsymbol{\beta}(s)^{-1}$.

We study system (1.2) which is based on the following points:

(1) (1.2) is one parameter deformation of (1.1). Deformation theory is a very important field in singularity theory and bifurcation theory, and has many applications in science and engineering (see [10] and [11]).

(2) (1.2) is equivariant under the action of Lie group

$$\{\mathrm{Ad}_{\boldsymbol{\beta}(s)} \mid s \in R_1\} : \mathrm{Ad}_{\boldsymbol{\beta}(s)} \circ [\boldsymbol{B},\ \boldsymbol{L}]_{\boldsymbol{\beta}(s)} = [\mathrm{Ad}_{\boldsymbol{\beta}(s)}(\boldsymbol{B}),\ \mathrm{Ad}_{\boldsymbol{\beta}(s)}(\boldsymbol{L})]_{\boldsymbol{\beta}(s)}.$$

This kind of differential equations is very important in equation theory and bifurcation theory (see [10] and [11]).

(3) For a Hom-Lie algebra $(\mathfrak{gl}(V),\ [\ \cdot\ ,\ \cdot\ ]_{\boldsymbol{\beta}(s)},\ \mathrm{Ad}_{\boldsymbol{\beta}(s)})$, when $\boldsymbol{\beta}(s) = \boldsymbol{I}_n$, it is just a Lie algebra $(\mathfrak{gl}(V),\ [\ \cdot\ ,\ \cdot\ ])$, where $\boldsymbol{I}_n$ is the $n \times n$ identity matrix.

The general framework is organized as follows: we first introduce the relevant definitions: one parameter deformation, $\Gamma$-equivariant and so on; then, we give definitions of $\boldsymbol{\beta}(s)$ and prove that $\{\mathrm{Ad}_{\boldsymbol{\beta}(s)} \mid s \in R_1\}$ is a Lie group. Second, we give three kinds of one parameter deformation of (1.1). Then, we study these deformations respectively and give solutions. At last, some problems are given.

## 2 Preliminaries

We first give some definitions, one can find these definitions easily in [10] and [11].

**Definition 2.1** *An equation*

$$g(x,\ s) = 0,$$

*where $x$ is an unknown variable, and the equation depends on an parameter $s(\in \mathbf{R})$. For a fixed $s_0$, let $g_1(x) = g(x,\ s_0)$. Then we call $g(x,\ s)$ is one parameter deformation of $g_1(x)$.*

**Definition 2.2** *A smooth map $g : \mathbf{R}^n \times \mathbf{R} \longrightarrow \mathbf{R}^n$ is $\Gamma$-equivariant, if for any $\gamma \in \Gamma$, $\Gamma$ is a Lie group, we have*

$$g(\gamma x,\ s) = \gamma g(x,\ s),$$

*where $\gamma x$ is the Lie group $\Gamma$ action on $\mathbf{R}^n$.*