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Embedding Cartesian Product of Some Graphs in Books

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Abstract: The book embedding of a graph G consists of placing the vertices of G in a line called spine and assigning edges of the graph to pages so that the edges assigned to the same page do not intersect. The number of pages is the minimum number in which the graph can be embedded. In this paper, we study the book embedding of the Cartesian product $P_m \times S_n$, $P_m \times W_n$, $C_n \times S_m$, $C_n \times W_m$, and get an upper bound of their pagenumber.

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1 Introduction

Let G be a simple graph. A book consists of a spine that is just a line and some number of pages each of which is a half-plane with the spine as boundary. We say that a graph admints an n-book embedding if one can assign the edges to n pages and there exists a linear ordering of its vertices along the spine of a book such that the edges on the same page do not intersect. The pagenumber is a measure of the quality of a book embedding. The pagenumber $pn(G)$ or book thickness $bt(G)$ of a graph G is the smallest n such that G has an n-book embedding.

Book embedding was introduced by Kainen[1] and later investigated by Bernhart and Kainen[2]. In [3], some applications of this problem are presented, such as sorting with parallel stacks, single-row routing, fault-tolerant processor arrays and turing machine graphs. Determining the pagenumber of an arbitrary graph embedded in book is NP-complete even if the order of vertices on the spine is fixed (see [3] and [4]). Few results are known for

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particular families of graphs. The pagenumber of the complete graphs K_n is $\left\lceil \frac{n}{2} \right\rceil$ (see [2]). In [5] and [6], the pagenumber of the complete bipartite graphs $K_{m,n}$ has been shown. The pagenumber of regular graphs has been solved in [7]. In [8], determining that the pagenumber of generalized Petersen graph is three in some situations, which is the best possible. There are many other kinds of graphs and the number of pages is also known, for example: toroidal graphs (see [9]), incomplete hypercubes (see [10]), complete odd-partite graphs (see [11]) and strong product graphs (see [12]), etc. The pagenumber of oriented graphs such as oriented cycles and oriented trees has also been studied.

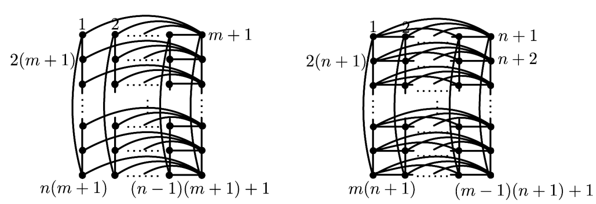
The pagenumber of many types of graphs which can be embedded into a given surface has also been investigated in some literatures. Heath and Istrail[13] showed that the graphs of genus g have pagenumber $O(g)$. Malitz[14] improved this to $O(\sqrt{g})$, which is a well bound. In [2], Bernhart and Kainen proved that: $pn(G) \leq 1$ if and only if G is outerplanar and $pn(G) \leq 2$ if and only if G is a subgraph of a hamiltonian planar graph. Hence, for a connected graph G which is neither an outplanar nor a subhamiltonian planar graph, we have $pn(G) \geq 3$.

In [12], the book embedding number of the Cartesian products $P_m \times P_n$, $P_m \times C_n$ has been given. In this article, we study the pagenumber of the Cartesian products $P_m \times S_n$, $P_m \times W_n$, $C_n \times S_m$, $C_n \times W_m$ and establish the upper bound of these graphs.

2 Main Results

The graph considered in this article are simple and connected. Given a graph G, with edge set $E(G)$ and vertex set $V(G)$, if two vertices i and j are adjacent in G, i.e., if there is an edge $e = ij \in E(G)$, we say that u and v are neighbors in G. we denote a path with m vertices and a cycle with n vertices by P_m and C_n, respectively. A wheel graph is a graph formed by connecting a single vertex to all vertices of a cycle and we use W_m to denote a wheel graph with $m+1$ vertices ($m \geq 3$). A star graph S_n with $n+1$ vertices is the complete bipartite graph $K_{1,n}$.

The Cartesian product of graphs G and H, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$, and $(u_1,\ v_1)$ is adjacent to $(u_2,\ v_2)$ if either u_1 is adjacent to u_2 in G and $v_1 = v_2$ or $u_1 = u_2$ and v_1 is adjacent to v_2 in H. And given a Cartesian product of path with star $P_m \times S_n$, there is a bijection $f\colon V \to \{1, 2, \cdots, m(n+1)\}$ from V to $[m(n+1)]$, so the edges v_iv_j can be replaced by $f(v_i)f(v_j) = (i,\ j)$. For example, the graph $C_n \times S_m$ (see Fig. 2.1(a)) and $P_m \times W_n$ (see Fig. 2.1(b)).



(a) (b)

Fig. 2.1 The graph $C_n \times S_m$ (left) and $P_m \times W_n$ (right)