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J-clean and Strongly J-clean Rings

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Abstract: Let R be a ring and $J(R)$ the Jacobson radical. An element a of R is called (strongly) J-clean if there is an idempotent $e \in R$ and $w \in J(R)$ such that $a = e + w$ (and $ew = we$). The ring R is called a (strongly) J-clean ring provided that every one of its elements is (strongly) J-clean. We discuss, in the present paper, some properties of J-clean rings and strongly J-clean rings. Moreover, we investigate J-cleanness and strongly J-cleanness of generalized matrix rings. Some known results are also extended.

Key words: J-clean ring, strongly J-clean ring, generalized matrix ring

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1 Introduction

Throughout this paper R is an associative ring with identity and all modules are unitary. We denote the Jacobson radical and the unit group of R by $J(R)$ and $U(R)$, respectively. We use $M_n(R)$ to stand for the ring of $n \times n$ matrices over a ring R.

An element a of a ring R is (strongly) clean provided that a is the sum of an idempotent e and a unit u in R (such that e and u commute). A ring R is (strongly) clean provided that every element in R is (strongly) clean. Clean rings were first defined by Nicholson (see [1]) as a class of exchange rings. It is well known that unit regular rings and semiperfect rings are also clean rings. The class of strongly clean rings was introduced in [2]. It was shown that all strongly π-regular rings are strongly clean and that a strongly clean endomorphism satisfies a generalized version of Fitting's lemma. In recent decades, many researchers studied such

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rings from various different perspectives (see [3]–[9]). Among the others, Chen[4] developed the concept of strongly J-clean rings to construct a subclass of strongly clean rings which have stable range one. Here an element $a \in R$ is strongly J-clean provided that there exists an idempotent e and an element $w \in J(R)$ such that $a = e + w$ and $ew = we$. The ring R is strongly J-clean if every element is strongly J-clean. It was proven that the ring of all 2×2 matrices over a commutative local ring is not strongly J-clean.

Following [10], given a ring R and a central element s of R, the 4-tuple $\begin{pmatrix} R & R \\ R & R \end{pmatrix}$ becomes a ring with addition defined componentwise and with multiplication defined by

$$\begin{pmatrix} a_1 & x_1 \\ y_1 & b_1 \end{pmatrix}\begin{pmatrix} a_2 & x_2 \\ y_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + sx_1y_2 & a_1x_2 + x_1b_2 \\ y_1a_2 + b_1y_2 & sy_1x_2 + b_1b_2 \end{pmatrix}.$$

This ring is denoted by $K_s(R)$. A Morita context is a 4-tuple $T := \begin{pmatrix} A & M \\ N & B \end{pmatrix}$, where A, B are rings, ${}_AM_B$ and ${}_BN_A$ are bimodules, and there exist context products $M \times N \to A$ and $N \times M \to B$ written multiplicatively as $(w,\ z) \mapsto wz$ and $(z,\ w) \mapsto zw$, such that T is an associative ring with the obvious matrix operations. A Morita context with $A = B = M = N = R$ is called a generalized matrix ring over R. It was observed by Krylov[10] that a ring S is a generalized matrix ring over R if and only if $S = K_s(R)$ for some $s \in C(R)$. Here $MN = NM = sR$, and so $MN \subseteq J(A)$ if and only if $s \in J(R)$. When $s = 1$, $K_1(R)$ is just the matrix ring $M_2(R)$, but $K_s(R)$ can be significantly different from $M_2(R)$. In [9], Tang and Zhou obtained necessary and sufficient conditions that $K_s(R)$ is strongly clean, where R is a general local ring and s is a central element in $J(R)$. As a consequence, a criterion was given for $K_s(R)$ to be strongly clean when R is a skew power series ring of a weakly bleached local ring. Further, for a commutative local ring R, criteria were obtained for a single element of $K_s(R)$ to be strongly clean.

In Section 2 of this paper, we study some basic properties of (strongly) J-clean rings. It is proven that strongly nil clean rings and quasipolar rings with some conditions are strongly J-clean. Section 3 is motivated to investigate J-cleanness and strongly J-cleanness of generalized matrix rings. We first show that a Morita context $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is J-clean if and only if A and B are J-clean and $MN \subseteq J(A)$, $NM \subseteq J(B)$. Thus, a matrix ring is never J-clean. Let R be a local ring with $s \in C(R)$ and $A \in K_s(R)$. We prove that A is strongly J-clean in $K_s(R)$ if and only if $A \in J(K_s(R))$ or $1 - A \in J(K_s(R))$ or A is similar to $\begin{pmatrix} 1 + w_1 & 0 \\ 0 & w_2 \end{pmatrix}$, where $w_1, w_2 \in J(R)$. If R is a commutative local ring and $A \in K_s(R)$, then A is strongly J-clean in $K_s(R)$ if and only if $A \in J(K_s(R))$ or $1 - A \in J(K_s(R))$ or the equation $x^2 - \operatorname{tr}(A)x + \det_s(A) = 0$ has a root in $J(R)$ and a root in $1 + J(R)$. Some results in [4] are extended.