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Existence and Uniqueness of Fixed Points for Mixed Monotone Operators with Application

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Abstract: In this paper, a class of mixed monotone operators is studied. Some new fixed point theorems are presented by means of partial order theory, and the uniqueness and existence of fixed points are obtained without assuming the operator to be compact or continuous. Our conclusions extend the relevant results. Moreover, as an application of our result, the existence and uniqueness of positive solution for a class of fractional differential equation boundary value problem are proved.

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1 Introduction and Preliminaries

In 2012, Borcut and Berinde[1] introduced the concept of tripled fixed point for nonlinear mappings in partially ordered complete metric spaces and obtained its existence.

In this paper, without assuming operators to be continuous or compact, we study tripled fixed point for a class of mixed monotone operator on ordered Banach spaces. This research done is important in comparison with others, as some times coupled fixed points are more perfect than other fixed points. Tripled fixed points are more practical than coupled fixed points and they are applicable for most differential equations which are not solve by the application of fixed points or coupled fixed points. As an application of our result, we

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prove the existence and uniqueness of a positive solution for a class of fractional differential equation boundary value problem which can not be solved by using previously available methods.

Let the real topological linear space E be partially ordered by a cone P of E, i.e., $x \leq y$ (or alternatively denoted as $y \geq x$) if and only if $y-x \in P$. We denote by θ the zero element of E. Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies

$$x \in P, \quad \lambda \geq 0 \Rightarrow \lambda x \in P,$$

and

$$x,\ -x \in P \Rightarrow x = \theta.$$

We denote by $\mathring{P}$ the interior set of P. A cone P is said to be solid cone if $\mathring{P} \neq \emptyset$. P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies

$$\|x\| \leq N\|y\|.$$

N is called the normal constant of P. We write $x \ll y$ if and only if $y - x \in \mathring{P}$.

Definition 1.1[2] *Let $D \subset E$. Operator $A\colon D \times D \to E$ is said to be mixed monotone if $A(x, y)$ is nondecreasing in x and nonincreasing in y, i.e., $u_1 \leq u_2$, $v_2 \leq v_1$, $u_i, v_i \in D$ $(i = 1, 2)$ implies $A(u_1, v_1) \leq A(u_2, v_2)$. Point $(x^*, y^*) \in D \times D$, $x^* \leq y^*$, is called a coupled lower-upper fixed point of A if $x^* \leq A(x^*, y^*)$ and $A(y^*, x^*) \leq y^*$. Point $(x^*, y^*) \in D \times D$ is called a coupled fixed point of A if $x^* = A(x^*, y^*)$ and $A(y^*, x^*) = y^*$. Element $x^* \in D$ is called a fixed point of A if $A(x^*, x^*) = x^*$.*

For all $x, y \in E$, the notation $x \sim y$ means that there exist $0 < \lambda \leq \mu$ such that $\lambda x \leq y \leq \mu x$. Clearly, "$\sim$" is an equivalence relation. Given $h > \theta$ (i.e., $h \geq \theta$ and $h \neq \theta$), we denote by P_h the set

$$P_h = \{x \in E,\ \text{there exist } \lambda(x),\ \mu(x) > 0 \text{ such that } \lambda(x)h \leq x \leq \mu(x)h\}.$$

It is easy to see that $P_h \subset P$.

On the product space $E^3 \doteq E \times E \times E$, consider the following partial order "$\prec$": for $(x, y, z),\ (u, v, w) \in E^3$,

$$(u, v, w) \prec (x, y, z) \Longleftrightarrow x \geq u,\ y \leq v,\ z \geq w.$$

Definition 1.2[1] *Let $(E, \leq)$ be a partially ordered set and $F\colon E^3 \to E$. We say that F has the mixed monotone property if for any $x, y, z \in E$,*

$$x_1, x_2 \in E,\ x_1 \leq x_2 \ \textit{ implies } \ F(x_1, y, z) \leq F(x_2, y, z);$$

$$y_1, y_2 \in E,\ y_1 \leq y_2 \ \textit{ implies } \ F(x, y_1, z) \geq F(x, y_2, z);$$

$$z_1, z_2 \in E,\ z_1 \leq z_2 \ \textit{ implies } \ F(x, y, z_1) \leq F(x, y, z_2).$$

Definition 1.3[1] *An element $(x, y, z) \in E^3$ is called a tripled fixed point of a mapping $F\colon E^3 \to E$ if $F(x, y, z) = x$, $F(y, x, y) = y$, $F(z, y, x) = z$.*

More concepts and facts about mixed monotone operators and other related concepts the reader is referred to [3]–[13] and some of the references therein.