Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.OA-2019-0073

Vol. **13**, No. 1, pp. 63-80
February 2020

Convergence Analysis of a Numerical Scheme for the Porous Medium Equation by an Energetic Variational Approach

Chenghua Duan[1,2], Chun Liu[3], Cheng Wang[4] and Xingye Yue[1,*]

[1] *Department of Mathematics, Soochow University, Suzhou 215006, China*
[2] *Shanghai Center for Mathematical Sciences, Fudan University, Shanghai 200438, China*
[3] *Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616, USA*
[4] *Department of Mathematics, University of Massachusetts, Dartmouth, North Dartmouth, MA 02747-2300, USA*

Received 11 May 2019; Accepted (in revised version) 1 September 2019

Abstract. The porous medium equation (PME) is a typical nonlinear degenerate parabolic equation. We have studied numerical methods for PME by an energetic variational approach in [C. Duan et al., J. Comput. Phys., 385 (2019), pp. 13–32], where the trajectory equation can be obtained and two numerical schemes have been developed based on different dissipative energy laws. It is also proved that the nonlinear scheme, based on $f \log f$ as the total energy form of the dissipative law, is uniquely solvable on an admissible convex set and preserves the corresponding discrete dissipation law. Moreover, under certain smoothness assumption, we have also obtained the second order convergence in space and the first order convergence in time for the scheme. In this paper, we provide a rigorous proof of the error estimate by a careful higher order asymptotic expansion and two step error estimates. The latter technique contains a rough estimate to control the highly nonlinear term in a discrete $W^{1,\infty}$ norm and a refined estimate is applied to derive the optimal error order.

AMS subject classifications: 65M12, 65M75

Key words: Energetic variational approach, porous medium equation, trajectory equation, optimal rate convergence analysis.

---

*Corresponding author. *Email addresses:* xyyue@suda.edu.cn (X. Y. Yue), chduan@fudan.edu.cn (C. H. Duan), cwang1@umassd.edu (C. Wang), cliu124@iit.edu (C. Liu)

1. Introduction and background

One of the typical nonlinear degenerate parabolic equations is the porous medium equation (PME):

$$\partial_t f = \Delta_x (f^m), \qquad x \in \Omega \subset \mathbb{R}^d, \qquad m > 1,$$

where $f = f(x,t)$ is a non-negative scalar function of space $x \in \mathbb{R}^d$ ($d \geq 1$) and the time $t \in \mathbb{R}^+$ and m is a constant larger than 1. It has been applied in many physical and biological models, such as an isentropic gas flow through a porous medium, the viscous gravity currents, nonlinear heat transfer and image processing [18], etc.

It is well known that the PME is degenerate at points where $f = 0$. In turn, the PME has many special features: the finite speed of propagation, the free boundary, a possible waiting time phenomenon [5, 18]. Various numerical methods have been studied for the PME, such as finite difference approach [8], tracking algorithm method [3], a local discontinuous Galerkin finite element method [24], Variational Particle Scheme (VPS) [23] and an adaptive moving mesh finite element method [13]. Many theoretical analyses have been derived in the existing literature [1, 12, 14, 16–18], etc.

Relevant detailed descriptions can be found in a recent paper [5], in which the numerical methods for the PME were constructed by an Energetic Variational Approach (EnVarA) to naturally keep the physical laws, such as the conservation of mass, energy dissipation and force balance. Meanwhile, based on different dissipative energy laws, two different numerical schemes have been studied. In more details, based on the total energy form $f \log f$ and $\frac{1}{2f}$, a fully discrete nonlinear scheme and a linear numerical scheme could be appropriately designed for the trajectory equation, respectively. It has also been proved that the former one is uniquely solvable on an admissible convex set and both schemes preserve the corresponding discrete dissipation law. Numerical experiments have demonstrated that both schemes have yielded a good approximation for the solution without oscillation and the free boundary. The notable advantage is that the waiting time problem could be naturally treated, which has been a well-known difficult issue for all the existing methods. In addition, under certain smoothness assumption, the second order convergence in space and the first order convergence in time have been reported for both schemes in [5]. The aim of the paper is to provide a rigorous proof of the optimal rate convergence analysis for the nonlinear scheme. On the other hand, the highly nonlinear nature of the trajectory equation makes the convergence analysis every challenging. To overcome these difficulties, we use a higher order expansion technique to ensure a higher order consistency estimate, which is needed to obtain a discrete $W^{1,\infty}$ bound of the numerical solution. Similar ideas have been reported in earlier literature for incompressible fluid equations [6, 7, 15, 21], while the analysis presented in this work turns out to be more complicated, due to the lack of a linear diffusion term in the trajectory equation of the PME. In addition, we have to carry out two step estimates to recover the nonlinear analysis:

Step 1 A rough estimate for the discrete derivative of numerical solution, namely ($D_h x_h^{n+1}$)