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Finite Difference/Element Method for a Two-Dimensional Modified Fractional Diffusion Equation

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Abstract. We present the finite difference/element method for a two-dimensional modified fractional diffusion equation. The analysis is carried out first for the time semi-discrete scheme, and then for the full discrete scheme. The time discretization is based on the $L1$-approximation for the fractional derivative terms and the second-order backward differentiation formula for the classical first order derivative term. We use finite element method for the spatial approximation in full discrete scheme. We show that both the semi-discrete and full discrete schemes are unconditionally stable and convergent. Moreover, the optimal convergence rate is obtained. Finally, some numerical examples are tested in the case of one and two space dimensions and the numerical results confirm our theoretical analysis.

AMS subject classifications: 26A33, 65M06, 65M12, 65M60

Key words: Modified subdiffusion equation, finite difference method, finite element method, stability, convergence rate.

1 Introduction

The time fractional derivative is a useful tool for modeling anomalous subdiffusion [17, 18], e.g., the time fractional diffusion equation

$$\frac{\partial u(\mathbf{x},t)}{\partial t} = \mu\, {}_0D_t^{1-\beta} \Delta u(\mathbf{x},t) + f(\mathbf{x},t), \quad \mathbf{x} \in \Omega, \quad t \in [0,T], \tag{1.1}$$

where Δ is the usual Laplace operator and $0 < \beta < 1$, μ is a positive constant; ${}_0D_t^{1-\beta}$ denotes the Riemann-Liouville fractional derivative of order $1-\beta$

$${}_0D_t^{1-\beta} v(t) = \frac{1}{\Gamma(\beta)} \frac{\partial}{\partial t} \int_0^t \frac{v(\tau)}{(t-\tau)^{1-\beta}} d\tau.$$

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For further investigating the less anomalous subdiffusion behavior of diffusion processes, a modified time fractional diffusion equation was proposed by introducing a secondary time fractional derivative acting on the diffusion operator [11, 21, 22]

$$\frac{\partial u(\mathbf{x},t)}{\partial t} = \big(\mu_0 D_t{}^{1-\beta} + \nu_0 D_t{}^{1-\gamma}\big)\Delta u(\mathbf{x},t) + f(\mathbf{x},t), \quad \mathbf{x}\in\Omega, \quad t\in[0,T], \tag{1.2}$$

where $0 < \beta, \gamma \leq 1$, μ and ν are positive constants. The quantity u is defined as a concentration or probability density function for the particles suspended in the liquid on a bounded domain Ω. For the particles described by (1.2), the relation between the mean square displacement of $x(t)$ of the diffusion particles and the time t is

$$\langle x^2(t)\rangle = \frac{2d\mu}{\Gamma(\beta+1)}t^{\beta} + \frac{2d\nu}{\Gamma(\gamma+1)}t^{\gamma}, \tag{1.3}$$

instead of

$$\langle x^2(t)\rangle = \frac{2d\mu}{\Gamma(\beta+1)}t^{\beta},$$

corresponding to (1.1). In (1.3), the mean square displacement of $x(t)$ is dominated by larger power for short times while for longer times it is dominated by the smaller power.

There are already some important progresses for the numerical solutions of the one-dimensional case of the time fractional diffusion equation (1.1), e.g., the finite difference method [4, 6, 10, 23–25]; Lin and Xu discuss the spectral method [14] with the convergence rate $\mathcal{O}(\tau^{2-\beta} + \tau^{-1}N^{-m})$, and Jiang and Ma analyze the finite element method [9] and show that the optimal convergent rate $\mathcal{O}(\tau^{2-\beta} + N^{-m})$ can be obtained, where m measures the regularity of the solution in space. Liu et al. study the finite element method for the one-dimensional case of (1.2) [16] with the convergent rate $\mathcal{O}(\tau + \tau^{-1}N^{-m})$. Here we further discuss the finite element method for (1.2) by using the $L1$ approximation [5, 14] to discretize the time fractional derivatives and show that the optimal convergent rate $\mathcal{O}(\tau^{1+\min\{\beta,\gamma\}} + N^{-m})$ is obtained. Instead of designing the numerical scheme straightforwardly, we first transform the Eq. (1.2) into

$$\begin{aligned}\frac{\partial u(\mathbf{x},t)}{\partial t} = {} & \big(\mu D_*^{1-\beta} + \nu D_*^{1-\gamma}\big)\Delta u(\mathbf{x},t) + \mu\frac{\Delta u(\mathbf{x},0)}{\Gamma(\beta)t^{1-\beta}} \\ & + \nu\frac{\Delta u(\mathbf{x},0)}{\Gamma(\gamma)t^{1-\gamma}} + f(\mathbf{x},t), \qquad \mathbf{x}\in\Omega, \quad t\in[0,T],\end{aligned} \tag{1.4}$$

where the relation between the Caputo fractional derivative and the Riemann-Liouville fractional derivative is used, given as [19]

$${}_0D_t^{1-\vartheta}v(t) = D_*^{1-\vartheta}v(t) + \frac{v(0)}{\Gamma(\vartheta)t^{1-\vartheta}}, \quad 0 < \vartheta \leq 1,$$