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A FOURTH-ORDER COMPACT AND CONSERVATIVE DIFFERENCE SCHEME FOR THE GENERALIZED ROSENAU-KORTEWEG DE VRIES EQUATION IN TWO DIMENSIONS*

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Abstract

In this paper, a conservative difference scheme for the Rosenau-Korteweg de Vries (RKdV) equation in 2D is proposed. The system satisfies the conservative laws in energy and mass. Existence and uniqueness of its difference solution have been shown. The order of $O(\tau^2+h^4)$ in the discrete L^∞-norm with time step τ and mesh size h is obtained. Some important lemmas are proposed to prove the high order convergence. We prove that the present scheme is unconditionally stable. Numerical results are also given in order to check the properties of analytical solution.

Mathematics subject classification: 65N06, 65M12, 65N22
Key words: RKdV equation, Conservation, Existence, Uniqueness, Stability, Convergence.

1. Introduction

There are mathematical models which describe the dynamics of wave behaviors, such as the KdV equation, the Rosenau equation, and many others. The existence and uniqueness of the solution for the Rosenau equation were proved by Park [1,2]. For the further consideration of nonlinear waves, the viscous term u_{xxx} needs to be included in the equation. This equation is usually called the Rosenau-KdV equation [3–8]. In this paper, we consider the 2D RKdV equation with initial and periodic boundary conditions as follows:

$$u_t+\Delta^2 u_t+\Delta u_x+(1+u^p)(u_x+u_y)=0, \qquad (x,y)\in\Omega, \quad t\in(0,T], \tag{1.1}$$

with periodic boundary condition

$$u(x,y,t)=u(x+L_1,y,t), \quad u(x,y,t)=u(x,y+L_2,t), \qquad (x,y)\in\Omega, \quad t\in(0,T], \tag{1.2}$$

and initial condition

$$u(x,y,0)=u_0(x,y), \qquad (x,y)\in\Omega, \tag{1.3}$$

where $\Omega=(0,L_1)\times(0,L_2)$, $\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$, $p\geq 1$ is an integer and $u_0(x,y)$ is (L_1,L_2)-periodic real function. In fact, the problem (1.1)-(1.3) is known to satisfy the following conservative laws (see [9]):

$$Q(t)=\iint_\Omega u(x,y,t)dxdy=\iint_\Omega u_0(x,y)dxdy=Q(0), \tag{1.4}$$

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and the energy

$$E(t) = \| u \|^2_{L^2(\Omega)} + \| \Delta u \|^2_{L^2(\Omega)} = E(0). \tag{1.5}$$

Due to uneasy control of the nonlinear term, numerical study of the Rosenau equation in 2D by the finite difference method is relative less. In [3], a conservative three-level linear finite difference scheme for the Rosenau-KdV equation is proposed. In [10], a new conservative finite difference scheme with real parameter for the Rosenau equation is given. In [9], Atouani and Omrani proposed two second-order conservative finite difference schemes for the RKdV equation in 2D. In [11], approximate solutions are considered for the extended Fisher-Kolmogorov (EFK) equation in two space dimension with Dirichlet boundary conditions by a Crank-Nicolson type finite difference scheme. Wang et al. proposed a high order compact multisymplectic scheme for coupled nonlinear Schrödinger-KdV equations [12].

Recently, there has been growing interest in high-order compact methods for solving partial differential equations (PDEs). Due to the convenience of implementation on machines, difference schemes are popular for Schröinger problems. The compact scheme [13–18] such as split-step schemes, implicit schemes, high-order schemes, unconditional convergence schemes and conservative schemes. The conservative numerical scheme for Rosenau-RLW equation were discussed in [19–22]. In [23–25], the high-order compact schemes where used for three-dimensional convection-diffusion equations, Sine-Gordon equation, heat and advection-diffusion equations. Unlike some previous techniques, using various transformations to reduce the equation into more simple equation and then solve it, the nonlinear equations are solved easily without transforming the equation by using the current method. This method has also additional advantages over some rival techniques, mainly, avoidance of linearization, ease in use, and computationally cost effective to find solutions of the given nonlinear equations. However, because the discretization of nonlinear term in compact scheme is more complicated than that in second-order one, a priori estimate in the discrete L^∞-norm is hard to be obtained, so the unconditional convergence of any compact difference scheme for nonlinear equation is difficult to be proved.

The remainder of this paper is arranged as follows. In Section 2, some notations are given and a new difference scheme is proposed. Some auxiliary lemmas are introduced or proved. In Section 3, unique solvability, discrete conservative laws of the proposed scheme are discussed. A priori estimate is obtained. In Section 4, the convergence is proved based on the estimation. The order of $O(\tau^2 + h^4)$ in the discrete L^∞-norm with time step τ and mesh size h is obtained. Lastly, numerical experiments are presented in Section 5.

2. The Compact Finite Difference Scheme

Let $h_1 = L_1/J_1$ and $h_2 = L_2/J_2$ be the space steps in the x and y directions respectively, where J_1, J_2 are any positive integers. And $\tau = T/N$ $(N \in \mathbf{N}^+)$ is the time step. We denote $U^n_{i,j}$ to be the numerical approximation of $u^n_{i,j} = u(x_i, y_j, t_n)$, where $x_i = ih_1, y_j = jh_2, 0 \le i \le J_1, 0 \le j \le J_2$, and $t_n = n\tau, 0 \le n \le N$. It can be seen from periodicity that $u^n_{0,j} = u^n_{J_1,j}, u^n_{i,0} = u^n_{i,J_2}$ and so on. Let the discrete grid $\Omega_h := \{(x_i, y_j) |\ 0 \le i \le J_1, 0 \le j \le J_2\}$, and

$$\mathbf{R}^{J_1,J_2}_{per} = \Big\{ V_{i,j} \in \mathbf{R} \Big|\ V_{0,j} = V_{J_1,j},\ V_{i,0} = V_{i,J_2},\ 0 \le i \le J_1,\ 0 \le j \le J_2 \Big\}.$$