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Locally Divergence-Free Spectral-DG Methods for Ideal Magnetohydrodynamic Equations on Cylindrical Coordinates

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Abstract. In this paper, we propose a class of high order locally divergence-free spectral-discontinuous Galerkin (DG) methods for three dimensional (3D) ideal magnetohydrodynamic (MHD) equations on cylindrical geometry. Under the conventional cylindrical coordinates (r,φ,z), we adopt the Fourier spectral method in the φ-direction and discontinuous Galerkin (DG) approximation in the (r,z) plane, motivated by the structure of the particular physical flows of magnetically confined plasma. By a careful design of the locally divergence-free set for the magnetic filed, our spectral-DG methods are divergence-free inside each element for the magnetic field. Numerical examples with third order strong-stability-preserving Runge-Kutta methods are provided to demonstrate the efficiency and performance of our proposed methods.

AMS subject classifications: 65M60, 65M70, 76W05

Key words: Discontinuous Galerkin method, magnetohydrodynamics (MHD), divergence-free, cylindrical coordinates.

1 Introduction

The ideal magnetohydrodynamics (MHD) equations are fluid models of perfectly conducting quasi-neutral plasmas, which have been widely used in astrophysics, space physics and plasma applications. Mathematically, the ideal MHD equations consist of

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nonlinear hyperbolic conservation laws for the macroscopic quantities with an additional divergence-free restriction on the magnetic field, which take the following form

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\mathbf{u}\\ \mathcal{E}\\ \mathbf{B}\end{bmatrix}+\nabla\cdot\begin{bmatrix}\rho\mathbf{u}\\ \rho\mathbf{u}\otimes\mathbf{u}+(p+\frac{1}{2}\|\mathbf{B}\|^2)\mathbb{I}-\mathbf{B}\otimes\mathbf{B}\\ \mathbf{u}(\mathcal{E}+p+\frac{1}{2}\|\mathbf{B}\|^2)-\mathbf{B}(\mathbf{u}\cdot\mathbf{B})\\ \mathbf{u}\otimes\mathbf{B}-\mathbf{B}\otimes\mathbf{u}\end{bmatrix}=0, \tag{1.1}$$

$$\nabla\cdot\mathbf{B}=0, \tag{1.2}$$

with

$$\mathcal{E}=\frac{p}{\gamma-1}+\frac{\rho\|\mathbf{u}\|^2}{2}+\frac{\|\mathbf{B}\|^2}{2}. \tag{1.3}$$

Here, ρ is density of mass, $\rho\mathbf{u}$ is momentum, $\mathcal{E}$ is total energy, p is the hydrodynamic pressure, $\|\cdot\|$ is used to denote the Euclidean vector norm and γ is the ideal gas constant. The ideal MHD equations (1.1) consist of eight coupled partial differential equations which are usually not solvable analytically.

Directly simulating the full 3D MHD system is in general very difficult due to the high demand on computational resources. However, the computational cost can be reduced for certain problems when considering geometric symmetry or structures. MHD computations regarding the stable equilibrium of magnetically confined plasma usually take place on cylindrical or toroidal geometry [6,42] and can be solved with cylindrical coordinates efficiently. It is known that for some physical problems such as the investigation of static equilibria and their perturbations, an important flow feature is toroidal symmetry or slow changes in the toroidal direction such as tokamak. In such cases, it is reasonable to assume that the radial component of the flow is smoother in the φ direction while the other components may involve large jumps [1,22,39]. Starting with this knowledge, dimension decomposition or dimension reduction can be performed to simplify the 3D MHD computation [17,23,25,32,40]. Here, we propose to use a similar approach to adopt the Fourier spectral method in the φ direction with a few modes and the DG approximation on the (r,z)-plane. By doing so, the 3D MHD computation is simplified to lower dimensions, and the computational cost is significantly reduced, which makes the numerical implementation more practical.

Another challenge in designing numerical methods for the ideal MHD equation is the divergence-free condition. That is, if the divergence of the initial magnetic field is zero, the divergence of the exact magnetic field at any future time is also zero. Honoring this divergence-free property for the numerical methods not only keeps consistency with the analytical conclusion but also reduces possible numerical instability that may cause break-down of the simulation. There are a lot of developments on divergence-free numerical methods for MHD equations in the literature, and those can be mainly categorized into four general approaches, i.e. the 8-wave formulation [21,29,33,34], the projection