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The Semi-Algebraic Split Feasibility Problem and its Semidefinite Relaxation

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Abstract. This paper considers the semi-algebraic split feasibility problem (SASFP), i.e., the split feasibility problem defined by polynomials. It is more than a special case of the split feasibility problem (SFP) or the multiple-sets split feasibility problem (MSFP), since the solution set could be nonconvex or empty. We first establish the semi-definite relaxation for the SASFP, then discuss on the relationship of feasibility between the SASFP and its SDP relaxation, especially focus on infeasibility. Finally, some numerical experiments for different cases are implemented, and the corresponding results are reported.

AMS subject classifications: 90C30, 90C25

Key words: Split feasibility problem, multiple-sets split feasibility problem, polynomial, semi-algebraic, semi-definite relaxation.

1. Introduction

The split feasibility problem (SFP) and the multiple-sets split feasibility problem (MSF-P) are useful models for many different practice problems. They arose from the fields of image reconstruction, signal processing and intensity-modulated radiation therapy, and so on (see, e.g., [2–4, 6], etc), as a unified model. In 1994, the SFP was first introduced by Censor and Elfving [4] to model phase retrieval problems. Later in 2005, a generalized model, i.e., the multiple-sets split feasibility problem (the MSFP) was proposed by Censor et al in [5].

The SFP is to find $x^* \in C$ such that $Ax^* \in Q$, where A is an m by n real matrix, and C and Q are nonempty closed convex sets in $\mathscr{R}^n$ and $\mathscr{R}^m$, respectively. Here, $\mathscr{R}^d$ denotes d-dimensional Euclidean space. The MSFP is to find a vector x^* satisfying

$$x^* \in \bigcap_{i=1}^{r} C_i, \quad \text{such that} \quad Ax^* \in \bigcap_{j=1}^{t} Q_j,$$

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where A is an $m \times n$ real matrix, $C_i \subseteq \mathscr{R}^n,\ i = 1, \cdots, r$ and $Q_j \subseteq \mathscr{R}^m,\ j = 1, \cdots, t$ are nonempty closed convex sets. If $t = r = 1$, then the MSFP reduces to the SFP. Also, if we take $C := \bigcap_{i=1}^{r} C_i$ and $Q := \bigcap_{j=1}^{t} Q_j$, then the SFP is actually an MSFP.

In decades of years, projection-type methods have been widely studied for solving the SFP and MSFP (see e.g. [2, 3, 5–10, 15, 19, 20, 23], etc). Some of these methods need to compute projections onto the convex sets C and Q, or C_i and Q_j; some need to compute projections onto half space containing the corresponding set; some have fixed iteration step; while some have self-adaptive iteration step. Among them, the CQ algorithm is most typical, and many other methods are actually variants and modifications based on it. However, convergence of these projection-type methods requires all the sets C, Q in the SFP, and C_i and Q_j in the MSFP are convex. However, it is well-known that many problems from practice are nonconvex. Thus we need to study on methods for solving the nonconvex SFP and MSFP.

It is known that a closed semialgebraic set is subset of $\mathscr{R}^n$ defined by a finite sequence of polynomial equations and inequalities, or any finite union of such sets (see e.g. [12,13]).

In this paper, we focus on the case that the SFP problem is defined by semi-algebraic sets:

let C and Q be nonempty sets defined as

$$C := \left\{x \in \mathscr{R}^n |\ f_i(x) \geq 0,\ i = 1, \cdots, r\right\}, \quad Q := \left\{y \in \mathscr{R}^m |\ g_j(y) \geq 0,\ j = 1, \cdots, t\right\},$$

where $f_i(x),\ g_j(y),\ i = 1, \cdots, r,\ j = 1, \cdots, t$ are multivariate polynomials. We are to find $x^* \in C$ such that $Ax^* \in Q$, where A is an m by n real matrix.

Note that this problem is defined by semi-algebraic sets, so we call it *semi-algebraic split feasibility problem (the SASFP, in short)*. This kind of problem is more than a special case of the SFP or the MSFP (with the sets $C_i := \{x \in \mathscr{R}^n |\ f_i(x) \geq 0\}$ and $Q_j := \{y \in \mathscr{R}^m |\ g_j(y) \geq 0\}$), because the sets C, Q, C_i and Q_j may or may not be convex in this kind of problems.

Notations. Through out this paper, the symbol $\mathscr{N}$ stands for the set of nonnegative integers, and $\mathscr{R}$ for the set of real numbers. For any $s \in \mathscr{R}$, $\lceil s \rceil$ denotes the smallest integer no smaller than s. For $x \in \mathscr{R}^n$, x_i denotes the i-th component of x. When an n-dimensional vector y is indexed by an integer vector $\alpha \in \mathscr{N}^n$, y_α denotes the entry of y whose index is α, and we also denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For $x \in \mathscr{R}^n$, x^α denotes $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

The rest of this paper is organized as follows: in Section 2, we establish the semi-definite relaxation for the SASFP; then in Section 3, we study on the relationship between the SASFP and its SDP relaxation, especially focus on infeasibility. In Section 4, some preliminary numerical experiments are executed, and the computing results are reported. Finally, in Section 5, some conclusions are stated.

2. Semidefinite relaxation of the semi-algebraic split feasibility problem

In this section, we are to establish semi-definite relaxation (called SDP relaxation, in short) for the above SASFP. The lifted technique in [13] is applied here.