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Fully Discrete H^1-Galerkin Mixed Finite Element Methods for Parabolic Optimal Control Problems

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Abstract. In this paper, we investigate a priori and a posteriori error estimates of fully discrete H^1-Galerkin mixed finite element methods for parabolic optimal control problems. The state variables and co-state variables are approximated by the lowest order Raviart-Thomas mixed finite element and linear finite element, and the control variable is approximated by piecewise constant functions. The time discretization of the state and co-state are based on finite difference methods. First, we derive a priori error estimates for the control variable, the state variables and the adjoint state variables. Second, by use of energy approach, we derive a posteriori error estimates for optimal control problems, assuming that only the underlying mesh is static. A numerical example is presented to verify the theoretical results on a priori error estimates.

AMS subject classifications: 49J20, 65N30

Key words: Parabolic equations, optimal control problems, a priori error estimates, a posteriori error estimates, H^1-Galerkin mixed finite element methods.

1. Introduction

Finite element method is the most widely used numerical method in computing optimal control problems, the literature on this topic is huge, it is impossible to even give a very brief review here. For the studies about a priori error estimates, superconvergence and a posteriori error estimates of finite element approximations for optimal control problems, see [2, 3, 7, 11, 15, 16, 19, 20, 22, 23, 31, 32] for elliptic optimal control problems and [12, 14, 18, 21, 24–26] for parabolic optimal control problems.

However, the mixed finite element method is much more important for a certain class of optimal control problems, which contains the gradient of the state variable in the objective functional. For example, in the flow control problem, the gradient stands for Dracy

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velocity and it is an important physics variable, or, in the temperature control problem, large temperature gradients during cooling or heating may lead to its destruction. Chen et al. have done some works on a priori error estimates and superconvergence properties of standard mixed finite element methods for optimal control problems, see, for example, [5, 6, 8, 13]. In [5, 6], Chen used the postprocessing projection operator, which was defined by Meyer and Rösch (see [22]) to prove a quadratic superconvergence of the control by mixed finite element methods. In [8], Chen used the average L^2 projection operator and the superconvergence properties of mixed finite element methods for elliptic problems to derive the superconvergence of the control. However, the convergence order is $h^{\frac{3}{2}}$ since the analysis was restricted by the low regularity of the control. In [13], we developed a mixed discontinuous finite element method for linear parabolic optimal control problems, and derived a priori and a posteriori error estimates.

In this paper, we shall investigate a priori and a posteriori error estimates of H^1-Galerkin mixed finite element method for parabolic optimal control problems. The proposed method was first introduced to discuss a priori error estimates for linear parabolic and parabolic integro-differential equations [27,28]. A notable advantage of this approach is that the method not only overcomes the inf-sup condition but the approximating finite element spaces are also allowed to be of different polynomial degree. Notice that using this method, we can derive two approximations for the gradient of the primal state variable y, one is the numerical approximation solution $\boldsymbol{p}_h$, the other is the derivative of the approximation solution y_h.

We consider the following linear parabolic optimal control problems for the state variables $\boldsymbol{p}$, y, and the control u with control constraint:

$$\min_{u\in K\subset U}\left\{\frac{1}{2}\int_0^T\left(\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\|y-y_d\|^2+\|u\|^2\right)dt\right\} \tag{1.1a}$$

$$\begin{aligned}
&y_t(x,t)+\mathrm{div}\boldsymbol{p}(x,t)+\boldsymbol{\beta}(x)\cdot\nabla y(x,t)+c(x)y(x,t)\\
&=f(x,t)+u(x,t), && x\in\Omega, && t\in J, && (1.1b)\\
&\boldsymbol{p}(x,t)=-A(x)\nabla y(x,t), && x\in\Omega, && t\in J, && (1.1c)\\
&y(x,t)=0, && x\in\partial\Omega, && t\in J, && (1.1d)\\
&y(x,0)=y_0(x), && x\in\Omega, && && (1.1e)
\end{aligned}$$

where $\Omega\subset\mathbf{R}^2$ is a polygonal domain, $J=(0,T]$. Let K be a closed convex set in $U=L^2(J;L^2(\Omega))$, $f,y_d\in L^2(J;L^2(\Omega))$, $\boldsymbol{p}_d\in L^2(J;(L^2(\Omega))^2)$, $y_0\in H^1(\Omega)$, $\boldsymbol{\beta}\in(W^{1,\infty}(\Omega))^2$ and $0<c\in W^{1,\infty}(\Omega)$. We assume that the coefficient matrix $A(x)=(a_{ij}(x))_{2\times 2}\in W^{1,\infty}(\bar{\Omega};\mathbf{R}^{2\times 2})$ is a symmetric 2×2-matrix and there are constants $c_1,c_2>0$ satisfying for any vector $\mathbf{X}\in\mathbf{R}^2$, $c_1\|\mathbf{X}\|^2_{\mathbf{R}^2}\le\mathbf{X}^tA\mathbf{X}\le c_2\|\mathbf{X}\|^2_{\mathbf{R}^2}$. K is a set defined by

$$K=\Big\{u\in U:\ u(x,t)\ge 0,\quad \text{a.e. in }\Omega\times J\Big\}. \tag{1.2}$$

We also assume that the following coercivity condition holds:

$$c-\frac{1}{2}\nabla\cdot\boldsymbol{\beta}\ge a_0>0.$$