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Efficient and Stable Exponential Runge-Kutta Methods for Parabolic Equations

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Abstract. In this paper we develop explicit fast exponential Runge-Kutta methods for the numerical solutions of a class of parabolic equations. By incorporating the linear splitting technique into the explicit exponential Runge-Kutta schemes, we are able to greatly improve the numerical stability. The proposed numerical methods could be fast implemented through use of decompositions of compact spatial difference operators on a regular mesh together with discrete fast Fourier transform techniques. The exponential Runge-Kutta schemes are easy to be adopted in adaptive temporal approximations with variable time step sizes, as well as applied to stiff nonlinearity and boundary conditions of different types. Linear stabilities of the proposed schemes and their comparison with other schemes are presented. We also numerically demonstrate accuracy, stability and robustness of the proposed method through some typical model problems.

AMS subject classifications: 65M06, 65M22, 65Y20

Key words: Exponential Runge-Kutta method, explicit scheme, linear splitting, discrete fast Fourier transforms, Allen-Cahn equation.

1 Introduction

The exponential integrator method is one of the many methods for solving stiff differential equations, see [7] for a recent review. Due to their stable and high-order accuracy for time integration, the exponential integrator based schemes have attracted many researcher's interests [2–5, 9, 11, 13–15]. Especially, along with the development of fast and stable methods for computing or approximating the product of a matrix exponential

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function with a vector (see the review [12]), the exponential integrator methods have become quite efficient and effective in practice. In this paper, we study exponential Runge-Kutta methods for a class of parabolic equations of the following form:

$$\frac{\partial u}{\partial t}=D\Delta u-f(t,u), \quad \mathbf{x}\in\Omega, \quad t\in[t_0,t_0+T], \tag{1.1}$$

where Ω is an open rectangular domain in $\mathbb{R}^d$, $T>0$, and D is a positive diffusion coefficient. Equations of the above form can represent mathematical models in various applications like Allen-Cahn equations [1] or Ginzburg-Landau equations in phase transition modeling. In order to solve the equations like (1.1), some explicit exponential Runge-Kutta methods have been developed and their numerical analysis were carried out [2–8, 11–13]. In [4], the authors analyzed the modifications of the expoential time differencing schemes via complex contour integrations and illustrated that the contour integration could improve the stability of the time integration. In [6], the authors proved convergence of explicit exponential Runge-Kutta methods up to order four and also constructed some new schemes that do not suffer from order reduction. In [13], the conditional stability of exponential Runge-Kutta methods are analyzed and some sufficient conditions are given. In [11], a fifth-order explicit exponential Runge-Kutta method with eight stages was proposed and its convergence was proved for semilinear parabolic problems.

More recently, in [8], by incorporating the linear splitting techniques (examined in [3, 4]) into the multistep approximation with an analytic evaluations of time exponential integrations, explicit compact exponential time differencing multistep methods are proposed for semilinear parabolic equations. These methods combine the decompositions of compact spatial difference operators on a regular mesh and fast discrete Fourier transform techniques to improve computation efficiency. By integrating seamlessly these well-studied techniques, the proposed compact exponential multistep methods can avoid solving nonlinear systems but are still stable and efficient to numerically solve semilinear problems. Because the discretization matrix is irreversible in the cases of periodic and Neumann boundary conditions, it is generally not an easy job to develop fast exponential methods, however, the compact exponential multistep methods developed in [8] can be easily applied to the problems with boundary conditions of different types. In this paper, the exponential Runge-Kutta methods are employed to approximate the temporal integrals. Compared with the multistep approximations, the exponential Runge-Kutta methods can be self-started, and more importantly, they are much easier to be implemented for adaptive time step sizes, that often can help to further improve the overall computation efficiency. In addition, the exponential Runge-Kutta methods are more stable than the corresponding multistep approaches as shown in Section 4.

The structure of this paper is as follows. In Section 2, we present a general compact exponential time integration scheme with a linear splitting parameter. Then, in Section 3, with the Runge-Kutta approximations for temporal integration, we propose fast and stable exponential Runge-Kutta methods in the compact form. In Section 4, linear stabilities