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Multi-Symplectic Method for the Zakharov-Kuznetsov Equation

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Abstract. A new scheme for the Zakharov-Kuznetsov (ZK) equation with the accuracy order of $\mathcal{O}(\Delta t^2+\Delta x+\Delta y^2)$ is proposed. The multi-symplectic conservation property of the new scheme is proved. The backward error analysis of the new multi-symplectic scheme is also implemented. The solitary wave evolution behaviors of the Zakharov-Kunetsov equation is investigated by the new multi-symplectic scheme. The accuracy of the scheme is analyzed.

AMS subject classifications: 65D17

Key words: The Zakharov-Kuznetsov equation, multi-symplectic method, backward error analysis.

1 Introduction

The two-dimensional generalization of the KDV equation, or the ZK equation

$$u_t+\frac{1}{2}(u^2)_x+u_{xxx}+u_{xyy}=0 \tag{1.1}$$

was first derived by Zakharov and Kuznetsov (1974) [26] in three dimensional form to describe nonlinear ion acoustic waves in a magnetized plasma [13,16]

$$u_t+uu_x+u_{xxx}+(\Delta u)_x=0, \quad \Delta=\partial_x^2+\partial_y^2+\partial_z^2. \tag{1.2}$$

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A variety of physical phenomena, in the purely dispersive limit, are governed by this type of equation; for example, the Rossby waves in rotating atmosphere [22], and the isolated vortex of the drift waves in three dimensional plasma [21]. Although Eq. (1.1) is not even integrable, quite a lot is now known about its nonlinear wave and soliton solutions. Numerical and analytical results of Eq. (1.1) have been investigated in [14,15].

Recently, Chen [9], from the Preissman scheme for multi-symplectic equations, derived a multi-symplectic numerical scheme for the ZK equation that can be simplified to an implicit 36-point scheme. In this paper, we proposed a new multi-symplectic Euler-box scheme to solve the two-dimensional ZK equation.

The paper is organized as follows: in Section 2, the multi-symplectic structure for the ZK equation is introduced and we propose a new multi-symplectic scheme for the ZK equation and prove its discrete multi-symplectic conservation law. In Section 3, we implement the backward error analysis for the new multi-symplectic scheme of the ZK equation. In Section 4, the solitary wave behaviors of the ZK equation are investigated by the new multi-symplectic scheme and the accuracy of the scheme is analyzed. We finish the paper with conclusion remarks in Section 5.

2 A new multi-symplectic scheme for the ZK equation

Introducing the potential $\varphi_x = u$, Eq. (1.1) is equivalent to

$$\varphi_{xxt}+\varphi_x\varphi_{xx}+\varphi_{xxxx}+\varphi_{xxyy}=0. \tag{2.1}$$

Now, we introduce some variables: $u=\varphi_x$, $v=\varphi_{xx}$, $w=\varphi_{xy}$, $p=-\varphi_{xt}/2$.

According to the covariant De Donder-Weyi Hamilton function theories and the multi-symplectic concept introduced by Bridges [2–7,12], the ZK equation can be reformulated as a system of five first-order partial differential equations which can be written in the form:

$$M\partial_t z+K\partial_x z+L\partial_y z=\nabla_z S(z),\quad z=(p,u,\varphi,v,w)^T\in R^5, \tag{2.2}$$

where

$$M=\begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{2} & 0 & 0\\ 0 & -\frac{1}{2} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},\quad K=\begin{pmatrix} 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ -1 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},\quad L=\begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

and $S(z)=up-(v^2+w^2)/2-u^3/6$. For details, we refer to [7], $\nabla_z S(z)$ is the gradient of $S(z)$ with respect to the standard inner product on R^5. The system (2.2) is a Hamiltonian formulation of the ZK equation on a multi-symplectic structure, where $M,K,L\in R^{n\times n}$ are skew-symmetric matrices and $S(z): R^n\to R$ is a smooth function of the $z(x,y,t)$.