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Crank-Nicolson Quasi-Wavelet Based Numerical Method for Volterra Integro-Differential Equations on Unbounded Spatial Domains

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Abstract. The numerical solution of a parabolic Volterra integro-differential equation with a memory term on a one-dimensional unbounded spatial domain is considered. A quasi-wavelet based numerical method is proposed to handle the spatial discretisation, the Crank-Nicolson scheme is used for the time discretisation, and second-order quadrature to approximate the integral term. Some numerical examples are presented to illustrate the efficiency and accuracy of this approach.

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Key words: Parabolic Volterra integro-differential equation, unbounded spatial domain, quasi-wavelet, Crank-Nicolson method.

1. Introduction

Integro-differential equations are quite common in science and engineering — e.g. to describe porous viscoelastic behaviour with known fluctuations, or vibrations and dynamic populations. Various algorithms have been designed for the numerical solution of Volterra integro-differential equations — including finite element methods [1, 13–16], an orthogonal spline collection method [5] and finite difference methods. In particular, Xu discussed the numerical solution of a fractional diffusion equation by a finite difference scheme in time and a Legendre spectral scheme in space [10], Liu considered the numerical soluttion of the Rayleigh-Stokes problem involving a fractional derivative for a heated generalised second grade fluid [11], and Tang used the Crank-Nicolson scheme to approximate a partial integro-differential equation with a weakly singular kernel [12].

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In this article, we consider the numerical solution to the the following problem. Find $u(x,t)$ satisfying the Volterra integro-differential equation

$$\frac{\partial u(x,t)}{\partial t} + \int_0^t k(x,t-s)u(x,s)ds = \triangle u(x,t) + f(x,t)\,, \quad x \in R,\ t \in [0,T] \tag{1.1}$$

where $\triangle u = \partial^2 u/\partial x^2$, subject to the initial condition

$$u(x,0) = u_0(x), \quad x \in R\,, \tag{1.2}$$

and the boundary condition

$$u \to 0 \quad as\ |x| \to \infty\,, \tag{1.3}$$

when the function $f(x,t)$ and the kernel function $k(x,t)$ are assumed to be sufficiently smooth.

When Eq. (1.1) applies on unbounded domains, numerical solutions have been obtained by many authors. One approach is the artificial boundary method to convert unbounded domain to bounded domains — e.g. Ma [19] used finite elements in space and the discontinuous Galerkin time-stepping method in time to solve the reduced problem, and the artificial boundary method for the numerical solution of parabolic PDEs on unbounded domains was considered in Refs. [20, 21]. An algebraic mapping has also been applied to the problem (1.1)-(1.3) on bounded domains, associated with the Legendre collocation method [22]. Here we use the Crank-Nicolson scheme is for the time discretisation, and the quasi-wavelet based numerical method for the spatial discretisation. The quasi-wavelet method is an effective way to approach the unbounded domain problem, since it is easy to implement and its distinctive local property produces accurate results. The localisation property allows us to analyse the local characteristics of functions involved [23], and the wavelet can be expressed as a superposition of its orthogonal scaling function. Thus the quasi-wavelet method is a very powerful tool for solving many kinds of partial integro-differential equations arising in real problems [17, 18, 24]. Interested readers may also refer to Refs. [28, 29, 35] for more detail on the quasi-wavelet numerical method.

We present the quasi-wavelet theory in Section 2. Subsection 3.1 presents the time discretisation for (1.1)-(1.3) via the Crank-Nicolson method, and the quasi-wavelet spatial discretisation and numerical algorithms are discussed in Section 3.2. Some numerical examples and results are given in Section 4, and concluding remarks in Section 5.

2. Quasi-Wavelet Based Numerical Method

Before giving a brief description of the quasi-wavelet based numerical method, let us first introduce the concept of singular convolution that often arise in science and engineering [25, 26]. A singular convolution is defined in the context of distribution theory as

$$F(t) = (T * g)(t) = \int_{-\infty}^{\infty} T(t-x)g(x)dx\,, \tag{2.1}$$