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High Order Difference Schemes for a Time Fractional Differential Equation with Neumann Boundary Conditions

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Abstract. A compact finite difference scheme is derived for a time fractional differential equation subject to Neumann boundary conditions. The proposed scheme is second-order accurate in time and fourth-order accurate in space. In addition, a high order alternating direction implicit (ADI) scheme is also constructed for the two-dimensional case. The stability and convergence of the schemes are analysed using their matrix forms.

AMS subject classifications: 65M06, 65M12, 65M15, 35R11

Key words: Time fractional differential equation, Neumann boundary conditions, compact ADI scheme, weighted and shifted Grünwald difference operator, convergence.

1. Introduction

Fractional differential equations have become the focus of many studies, due to their various applications. Many problems from signal processing, anomalous diffusion and finance can be modelled more accurately using equations with fractional derivatives. For example, when studying universal electromagnetic responses involving the unification of diffusion and wave propagation phenomena, there are processes that are described by equations with time fractional derivatives of order between 1 and 2 (to "interpolate" between diffusion equations and wave equations). Reference may be made to the books [1,2] for further information, where theoretical results such as solution existence and uniqueness can also be found. One of the key features of fractional derivatives is nonlocal dependence, which causes difficulties when numerical schemes for solving fractional differential equations are designed, but substantial progress has been made in recent years – e.g. see Refs. [3–25], and in particular Refs. [21–25] where time fractional differential equations subject to Neumann boundary conditions are discussed.

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We consider high order finite difference schemes for the numerical solution in a region Ω for problems in the following form:

$$
\begin{aligned}
{}_0^C D_t^{\gamma} u(\mathbf{x},t) &= \kappa_1 \Delta u(\mathbf{x},t) - \kappa_2 u(\mathbf{x},t) + g(\mathbf{x},t)\,, \\
\mathbf{x} &\in \Omega\,, \qquad 0 < t \le T\,, \qquad 1 < \gamma < 2\,,
\end{aligned}
\tag{1.1}
$$

subject to the initial conditions

$$
u(\mathbf{x},0) = \psi(\mathbf{x})\,, \qquad \frac{\partial u(\mathbf{x},0)}{\partial t} = \phi(\mathbf{x})\,, \qquad \mathbf{x} \in \bar{\Omega} = \Omega \cup \partial\Omega\,,
$$

and the zero flux boundary condition

$$
\frac{\partial u(\mathbf{x},t)}{\partial \mathbf{n}} = 0\,, \qquad \mathbf{x} \in \partial\Omega\,, \qquad 0 < t \le T\,,
$$

where $\partial\Omega$ is the boundary of Ω, $\partial/\partial\mathbf{n}$ is the differentiation in the normal direction and κ_1, κ_2 are positive constants. Here ${}_0^C D_t^{\gamma} u$ denotes the Caputo fractional derivative of u with respect to the time variable t:

$$
{}_0^C D_t^{\gamma} u(\mathbf{x},t) = \frac{1}{\Gamma(2-\gamma)} \int_0^t \frac{\partial^2 u(\mathbf{x},s)}{\partial s^2} (t-s)^{1-\gamma} ds,
$$

where $\Gamma(\cdot)$ is the gamma function. Eq. (1.1) can be rewritten as (cf. [15])

$$
\begin{aligned}
\frac{\partial u(\mathbf{x},t)}{\partial t} = \phi(\mathbf{x}) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[\kappa_1 \frac{\partial^2 u(\mathbf{x},s)}{\partial x^2} - \kappa_2 u(\mathbf{x},s) \right] ds + f(\mathbf{x},t)\,, \\
\mathbf{x} \in \Omega\,, \quad 0 < t \le T\,,
\end{aligned}
$$

where $0 < \alpha = \gamma - 1 < 1$, $f(\mathbf{x},t) = {}_0 I_t^{\alpha} g(\mathbf{x},t)$, and ${}_0 I_t^{\alpha}$ is the Riemann-Liouville fractional integral operator of order α defined as

$$
{}_0 I_t^{\alpha} g(\mathbf{x},t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(\mathbf{x},s) ds\,.
$$

We also note that $\partial\psi(\mathbf{x})/\partial\mathbf{n} = 0$ for $\mathbf{x} \in \partial\Omega$.

By applying the weighted and shifted Grünwald difference (cf. [13, 14, 16]) to the Riemann-Liouville fractional integral, in this article we establish compact schemes with second-order temporal accuracy and fourth-order spatial accuracy. Our analysis is based on the matrix form of the schemes, which turns out to render some of the intuitive ideas on certain norms and inner products noted in previous related articles. In Section 2 and Section 3, we first consider the one-dimensional case of Eq. (1.1), where we propose a high order scheme and study its convergence. In Section 4, a high order alternating direction implicit scheme is then proposed for the two-dimensional case. Numerical examples are given in the last section.