East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.060317.020617a

Vol. **7**, No. 3, pp. 603-614
August 2017

Primal-Dual Active Set Method for American Lookback Put Option Pricing

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Received 6 March 2017; Accepted (in revised version) 2 June 2017.

Abstract. The pricing model for American lookback options can be characterised as a two-dimensional free boundary problem. The main challenge in this problem is the free boundary, which is also the main concern for financial investors. We use a standard technique to reduce the pricing model to a one-dimensional linear complementarity problem on a bounded domain and obtain a corresponding variational inequality. The inequality is discretised by finite differences and finite elements in the temporal and spatial directions, respectively. By enforcing inequality constraints related to the options using Lagrange multipliers, the discretised variational inequality is reformulated as a set of semi-smooth equations, which are solved by a primal-dual active set method. One of the major advantages of our algorithm is that we can obtain the option values and the free boundary simultaneously, and numerical simulations show that our approach is as efficient as some other methods.

AMS subject classifications: 35A35, 90A09, 65K10, 65M12, 65M60

Key words: American lookback option, linear complementarity problem, variational inequality, finite element method, primal-dual active set method.

1. Introduction

The valuation of financial derivatives is important in financial markets such as the commodity market or the stock market. Options are special derivatives used to insure investments against a market downturn. According to the exercise time, options can be divided into two categories — viz. European style and American style. Due to its early exercise possibility, the American style is more popular with investors, corporations, mutual funds, and financial institutions. Here we focus on valuation of complex and flexible American lookback options, which are path-dependent with payoff functions depending on the maximum or minimum of the underlying asset price over the life of the options. Compared with

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other options, the advantage of the lookback option model is twofold. On the one hand, it offers opportunities for investors whose concern is underlying asset price fluctuations during the life of the options but not the price on the maturity date. On the other hand, lookback options can be exercised at any time up to the maturity date as for other American options. However, in the absence of closed form solutions, the fast, efficient and reliable pricing methods for lookback options are in great demand.

Let S be the underlying asset price. In the classical framework of the Black-Scholes model [23, 26, 31], the payoff function of lookback options with a lifetime $[0, T]$ can be represented as

$$G(S,J,t) = \begin{cases} J - S & \text{for put option with } J = \max\limits_{0\le\varsigma\le t} S_\varsigma \\ S - J & \text{for call option with } J = \min\limits_{0\le\varsigma\le t} S_\varsigma \end{cases}, \quad 0 \le t \le T .$$

We consider lookback put option pricing problems — lookback call options can be treated similarly. Assume that the interest rate is a constant r, and let σ and q denote the volatility and dividend rate of S, respectively. Then the American lookback put option value $P(S,J,t)$ is the solution of the free boundary problem

$$\mathscr{L}P := P_t + \frac{\sigma^2}{2}S^2 P_{SS} + (r-q)SP_S - rP = 0\,, \quad B(t) < S \le J < +\infty\,, \ \ 0 \le t < T\,. \tag{1.1}$$

This model has been solved by various numerical methods [7,20,22]. The binomial method, used to price classical American options [4, 6], was generalised and applied to American lookback options [1, 18, 19]. The finite difference and finite element methods have also been used in pricing American lookback options [5, 8].

In the free boundary problem (1.1), it is notable that one has to determine the optimal exercise boundary $B(t)$ in advance. To avoid the resulting computational difficulties, the problem (1.1) can be reformulated as a linear complementarity problem [17]:

$$\begin{cases} \mathscr{L}P \cdot (P-G) = 0 \\ \mathscr{L}P \le 0\,,\ P \ge G \end{cases}, \quad 0 \le S \le J < +\infty\,, \ \ 0 \le t < T\,. \tag{1.2}$$

This formulation does not contain any information about the free boundary. Numerical methods such as the Projected Successive Over Relaxation (PSOR) [12], projection and contraction [24], penalty [32], Lagrangian [15] and augmented Lagrangian [28] methods have been developed for the linear complementarity problem. These strategies have also been applied to the valuation of American lookback options [25,27,30]. A finite difference method to solve the linear complementarity problem (1.2) for American lookback options has been considered, where the option prices are obtained by the PSOR method [30]. The projection and contraction method has also been considered for the discretised variational inequality corresponding to American lookback options [25].

Here we also consider the linear complementarity problem to value American lookback options. According to Ref. [30], the linear complementarity problem (1.2) can be expressed as

$$\begin{cases} \mathscr{L}P \cdot (P-G) = 0 \\ \mathscr{L}P \le 0\,,\ P \ge G \end{cases}, \quad JX \le S \le J < +\infty\,, \ \ 0 \le t < T\,, \tag{1.3}$$