East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.031116.080317a

Vol. **7**, No. 3, pp. 439-454
August 2017

Numerical Solution of the Time-Fractional Sub-Diffusion Equation on an Unbounded Domain in Two-Dimensional Space

Hongwei Li[1], Xiaonan Wu[2] and Jiwei Zhang[3,*]

[1] *School of Mathematics and Statistics, Shandong Normal University, Ji'nan 250014, China.*
[2] *Department of Mathematics, Hong Kong Baptist University, Hong Kong, China.*
[3] *Beijing Computational Science Research Center, Beijing 100193, China.*

Received 3 November 2016; Accepted (in revised version) 8 March 2017.

Abstract. The numerical solution of the time-fractional sub-diffusion equation on an unbounded domain in two-dimensional space is considered, where a circular artificial boundary is introduced to divide the unbounded domain into a bounded computational domain and an unbounded exterior domain. The local artificial boundary conditions for the fractional sub-diffusion equation are designed on the circular artificial boundary by a joint Laplace transform and Fourier series expansion, and some auxiliary variables are introduced to circumvent high-order derivatives in the artificial boundary conditions. The original problem defined on the unbounded domain is thus reduced to an initial boundary value problem on a bounded computational domain. A finite difference and L1 approximation are applied for the space variables and the Caputo time-fractional derivative, respectively. Two numerical examples demonstrate the performance of the proposed method.

AMS subject classifications: 65M06, 65M12, 65N06

Key words: Fractional sub-diffusion equation, unbounded domain, local artificial boundary conditions, finite difference method, Caputo time-fractional derivative.

1. Introduction

Time-fractional sub-diffusion equations describe anomalous diffusion in many complex systems such as porous materials, nuclear magnetic resonance, percolation clusters, and random and disordered media [1–6]. The time-fractional sub-diffusion equation also arises in continuous-time random walks with temporal memories, which is characterised by asymptotic behaviour of the mean-square displacement — i.e. $\langle r^2(t)\rangle \sim t^\alpha$ with an anomalous diffusion exponent $0 < \alpha < 1$ [3,6].

*Corresponding author. *Email addresses:* `hwli@sdnu.edu.cn` (H. Li), `xwu@hkbu.edu.hk` (X. Wu), `jwzhang@csrc.ac.cn` (J. Zhang)

The increasing importance of time-fractional sub-diffusion equations in many practical applications has attracted immense interest in recent years, and their solution on bounded domains has been widely studied analytically and numerically. However, exact solutions of most fractional differential equations have only been obtained in specific cases associated with Fox, Wright and Mittag-Leffler special functions [7–9]. On the other hand, there are many numerical methods to solve related initial or boundary value problems on bounded domains in space. One numerical approach for the one-dimensional fractional sub-diffusion equation with Caputo time-fractional derivatives used the finite difference method [10], and another effective difference scheme was based on an L1 approximation for the Caputo time-fractional derivative [11]. An implicit meshless approach was proposed for the time-fractional sub-diffusion equation in two space dimensions [12], and the alternating direction implicit scheme has also been considered [13–15]. A semi-discrete method has been invoked for a class of time-fractional diffusion equations describing tracer solute transport in an aquifer [16], where the numerical predictions were compared with experimental data. A second-order finite difference method for two-dimensional fractional percolation equations has also been designed [17], and some other solution methods for fractional sub-diffusion equations can be found in Refs. [18–23] and references therein.

Numerical solution of time-fractional sub-diffusion equations on unbounded domains has also received some attention. The artificial boundary method (ABM), said to be very general and applicable, involves truncating the unbounded domain around the region of interest by using an artificial boundary and designing a suitable boundary condition on the artificial boundary. The original problem defined on the unbounded domain is thus reduced to an initial boundary value (IBV) problem on a bounded computational domain, which is a good approximation to the original problem with a suitably chosen boundary condition on the artificial boundary. Recently, Gao & Sun [24] designed exact artificial boundary conditions for the one-dimensional time-fractional sub-diffusion equation, and constructed some finite difference schemes to solve the reduced IBV problem. Brunner *et. al* [25] constructed exact artificial boundary conditions for a time-fractional diffusion-wave equation on a two-dimensional unbounded domain, and obtained a series of approximate artificial boundary conditions with high accuracy. Moreover, the order of convergence and stability estimates for two finite difference schemes were also analysed rigorously. For the time-fractional sub-diffusion equation on two-dimensional unbounded domains in space, Ghaffari & Hosseini [26] derived exact and approximate artificial boundary conditions on a circular artificial boundary using the Laplace transform and a Fourier series expansion. The classical Crank-Nicolson method for space variables and L1 approximation for the Caputo time-fractional derivative were used to solve the reduced problem. Other exact and local artificial boundary conditions have also been studied [27–31].

Here we consider high-order local artificial boundary conditions (LABCs) for the time-fractional sub-diffusion equation on two-dimensional unbounded domains. As it is difficult to design the suitable high-order artificial boundary conditions at the corners of a rectangular domain, in Section 2 a circular artificial boundary $\Gamma_R = \{(r,\theta)|r = R, 0 \leq \theta < 2\pi\}$ with radius R is introduced to divide the unbounded domain into two parts — viz. the bounded computational domain $\Omega_R = \{(r,\theta)|r < R, 0 \leq \theta < 2\pi\}$ and the unbounded exte-