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Application of gPCRK Methods to Nonlinear Random Differential Equations with Piecewise Constant Argument

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Abstract. We propose a class of numerical methods for solving nonlinear random differential equations with piecewise constant argument, called gPCRK methods as they combine generalised polynomial chaos with Runge-Kutta methods. An error analysis is presented involving the error arising from a finite-dimensional noise assumption, the projection error, the aliasing error and the discretisation error. A numerical example is given to illustrate the effectiveness of this approach.

AMS subject classifications: 65L70, 65C30

Key words: Random differential equations, piecewise constant argument, generalised polynomial chaos, finite-dimensional noise, error analysis.

1. Introduction

In recent years, delay differential equations (DDEs)

$$\begin{cases} x'(t) = f\big(t, x(t), x(\lfloor t \rfloor)\big), & t \in [0, T], \\ x(0) = x_0 \end{cases} \tag{1.1}$$

with piecewise constant argument $\lfloor t \rfloor$ (where $\lfloor \cdot \rfloor$ denotes the greatest-integer function) have been considered quite widely — e.g. see Refs. [6, 7, 12–14] and references therein. In particular, the solvability, stability, oscillations, dissipativity and other related properties of the analytical and numerical solutions were investigated. However, previous research

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has focused on deterministic equations, and in this article we consider random differential equations with piecewise constant argument (RDEPCA) as follows:

$$\begin{cases} u'(t,\omega) = f\big(t, \gamma(t,\omega), u(t,\omega), u(\lfloor t \rfloor, \omega)\big), & t \in [0,T],\ \omega \in \Omega, \\ u(0) = \varphi_0 . \end{cases} \tag{1.2}$$

Here $(\Omega, \mathscr{F}, \mathscr{P})$ denotes a complete probability space with a sample space Ω, a σ-algebra $\mathscr{F} \subset 2^{\Omega}$ and a probability measure $\mathscr{P}$; f is a random function in the random field $\gamma(t,\omega)$ with an assigned distribution; and the initial value φ_0 is given. For convenience, $u(t,\omega)$ will be denoted by $u(t)$ hereafter. The regularity of the solution of the RDEPCA (1.2) can be discussed as in Ref. [18].

We combine generalised polynomial chaos (gPC) with Runge-Kutta (RK) methods, producing so-called gPCRK methods to solve the RDEPCA (1.2). The gPC method of Xiu *et al.* [17] has been applied successfully to some random models involving ordinary and partial differential equations [9,10,16]) — cf. also the review article [8,11,15]. A gPC method can change a random model into a deterministic one, so one can transform the RDEPCA (1.2) into a set of deterministic DDEs and then obtain its solution by applying a suitable RK method. In Section 2, we introduce a finite-dimensional noise assumption and gPC expansion, and then construct our class numerical methods for the RDEPCA (1.2) by combining the gPC with Runge-Kutta methods. In Section 3, the error analysis involves the error arising from the finite-dimensional noise assumption, the projection error, the aliasing error and the discretisation error. In Section 4, we give a numerical example to illustrate the effectiveness of the method, followed by our conclusions in Section 5.

2. Construction of the Numerical Methods

In order to describe the solution of the RDEPCA (1.2) by a finite number of random variables, we introduce a so-called finite-dimensional noise assumption — e.g. see Babuska *et al.* [1]. Thus a random parameter $\gamma(t,\omega)$ can be expressed approximately as

$$\gamma(t,\omega) \approx \gamma\big(t, \xi_1(\omega), \xi_2(\omega), \cdots, \xi_N(\omega)\big), \quad (t,\omega) \in [0,T] \times \Omega, \tag{2.1}$$

where $N \in \mathbb{Z}^+$, and $\{\xi_n\}_{n=1}^N$ are real-valued random variables with zero mean and unit variance.

Here we assume that the random variables $\{\xi_n\}_{n=1}^N$ are independent, so the random parameter $\gamma(t,\omega)$ admits a truncated series representation [4,16]:

$$\gamma(t,\omega) = \mathbb{E}\{\gamma\} + \sum_{n=1}^{\infty} \gamma_n(t)\xi_n(\omega) \approx \mathbb{E}\{\gamma\} + \sum_{n=1}^{N} \gamma_n(t)\xi_n(\omega) \triangleq \gamma_N(t,y) \tag{2.2}$$

where $y = (\xi_1(\omega), \xi_2(\omega), \cdots, \xi_N(\omega))$, with a joint probability density

$$\rho(y) = \prod_{i=1}^{N} \rho(\xi_i), \quad \Gamma = \prod_{i=1}^{N} \Gamma_i \to \mathbb{R}^+ \ \text{ where } \Gamma_i = \xi_i(\Omega),$$