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A PRACTICAL PARALLEL DIFFERENCE SCHEME FOR PARABOLIC EQUATIONS *

Jia-quan Gao
(*RDCPS, Institute of Software, Chinese Academy of Sciences, P.O.Box 8718, Beijing 100080, China*)

Abstract

A practical parallel difference scheme for parabolic equations is constructed as follows: to decompose the domain Ω into some overlapping subdomains, take flux of the last time layer as Neumann boundary conditions for the time layer on inner boundary points of subdomains, solve it with the fully implicit scheme on each subdomain, then take correspondent values of its neighbor subdomains as its values for inner boundary points of each subdomain and mean of its neighbor subdomain and itself at overlapping points. The scheme is unconditionally convergent. Though its truncation error is $O(\tau + h)$, the convergent order for the solution can be improved to $O(\tau + h^2)$.

Key words: Parallel difference scheme, Parabolic equation, Segment, Explicit–implicit.

1. Introduction

As we know, the drawback of pure explicit schemes for solving parabolic problems is a very restrictive constraint on a time step. Fully implicit schemes are unconditional stable, but their drawback is that on each time level, linear or nonlinear algebraic systems have to be solved, and it is not easy to parallel implementation. Therefore people wish to find a kind of difference schemes such that it has a lower restrictive constraint on a time step, and at best is unconditionally stable and convergent; the other hand, it has higher parallel character, that is, the solution domain can be decomposed freely, and loaded balance is convenient to be implemented, and only communication between neighbor CPU can be developed with few time. In this paper, we study this problem, and construct a practical parallel difference scheme for the following problem

$$\begin{cases} u_t = u_{xx}, & (x,t) \in \Omega & (1) \\ u(x,0) = \phi(x), & 0 \le x \le L & (2) \\ u(0,t) = u(L,t) = 0, & 0 \le t \le T & (3) \end{cases}$$

where $\Omega = [0,L] \times [0,T]$. For solving this equations, Zhou[1,2] presented some difference schemes with intrinsic parallelism, and proved their stability and convergence[3]. Zhang[4,5] also provided the alternating segment explicit–implicit scheme. In addition, there were still other alternating explicit–implicit schemes etc[6,7]. In this paper, a practical parallel difference scheme is presented, and can be extended to more general practical problems, including variable coefficient equations, or low order items, and two or more dimensional equations etc.

This paper is outlined as follows. In the second section, a practical parallel difference scheme and its convergence is proved. Some numerical experiments are presented in the third section.

2. Parallel Difference Scheme

2.1. Construction of the Parallel Difference Scheme

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Divide the domain$[0,L]\times[0,T]$ into small grids with the space steplength h and time steplength τ, where $Jh=L, N\tau=T$, and J and N are positive integers. Denote by $u_j^n(j=0,1,\cdots,J;n=0,1,\cdots,N)$ the discrete function defined on the discrete rectangular domain $\{(x_j,t^n)|j=0,1,\cdots,J;n=0,1,\cdots,N\}$ of the grid points, $\lambda=\frac{\tau}{h^2}$, $\Delta_\tau u_j^{n+1}=\frac{1}{\tau}(u_j^{n+1}-u_j^n)$, and $\delta^2 u_j^{n+1}=\frac{1}{h^2}(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1})$. Assume that we have obtained the values of the discrete function $\{u_j^n|j=1,2,\cdots,J-1\}$ on the nth time layer, and will take values $\frac{u_{k+2}^n-u_{k+1}^n}{h}$, $\frac{u_{k-2}^n-u_{k-1}^n}{h}$ of the nth time layer as Newmann boundary values on these boundary points of subdomains. For convenience, we only decompose the domain Ω into two subdomains, which are $[0,x_{k+2}]$ and $[x_{k-2},L]$. Thus, we can use the implicit scheme inside the two subdomain, and take the values of the $(n+1)$th time layer at these points $x=x_{k-1}, x=x_k$ and $x=x_{k+1}$ as estimating values, Denoting them by $\bar{u}_j^{n+1}(j=k-1,k,k+1)$ and $\tilde{u}_j^{n+1}(j=k-1,k,k+1)$ on two subdomains $[0,x_{k+2}]$ and $[x_{k-2},L]$ respectively. The difference scheme is written as follows

$$\Delta_\tau u_j^{n+1}=\delta^2 u_j^{n+1},\quad j=1,2,\cdots,k-3,k+3,\cdots,J-1, \tag{4}$$

$$\Delta_\tau u_{k-2}^{n+1}=\frac{1}{h^2}(\bar{u}_{k-1}^{n+1}-2u_{k-2}^{n+1}+u_{k-3}^{n+1}), \tag{5}$$

$$\Delta_\tau \bar{u}_{k-1}^{n+1}=\frac{1}{h^2}(u_{k-2}^{n+1}-2\bar{u}_{k-1}^{n+1}+\bar{u}_k^{n+1}), \tag{6}$$

$$\Delta_\tau \bar{u}_k^{n+1}=\frac{1}{h^2}(\bar{u}_{k+1}^{n+1}-2\bar{u}_k^{n+1}+\bar{u}_{k-1}^{n+1}), \tag{7}$$

$$\Delta_\tau \bar{u}_{k+1}^{n+1}=\frac{1}{h^2}(u_{k+2}^n-u_{k+1}^n-\bar{u}_{k+1}^{n+1}+\bar{u}_k^{n+1}), \tag{8}$$

$$\Delta_\tau u_{k+2}^{n+1}=\frac{1}{h^2}(u_{k+3}^{n+1}-2u_{k+2}^{n+1}+\tilde{u}_{k+1}^{n+1}), \tag{9}$$

$$\Delta_\tau \tilde{u}_{k+1}^{n+1}=\frac{1}{h^2}(u_{k+2}^{n+1}-2\tilde{u}_{k+1}^{n+1}+\tilde{u}_k^{n+1}), \tag{10}$$

$$\Delta_\tau \tilde{u}_k^{n+1}=\frac{1}{h^2}(\tilde{u}_{k-1}^{n+1}-2\tilde{u}_k^{n+1}+\tilde{u}_{k+1}^{n+1}), \tag{11}$$

$$\Delta_\tau \tilde{u}_{k-1}^{n+1}=\frac{1}{h^2}(u_{k-2}^n-u_{k-1}^n-\tilde{u}_{k-1}^{n+1}+\tilde{u}_k^{n+1}), \tag{12}$$

and then we take

$$u_{k-1}^{n+1}=\bar{u}_{k-1}^{n+1},\ u_{k+1}^{n+1}=\tilde{u}_{k+1}^{n+1},\ u_k^{n+1}=\frac{1}{2}(\bar{u}_k^{n+1}+\tilde{u}_k^{n+1}). \tag{13}$$

Obviously this difference scheme is solved segmentally. Eqs.(4)$\sim$(8) is one segment, and Eqs.(4), (9) $\sim$ (12) is another segment. Both Eqs.(4)$\sim$(8) and Eqs.(4),(9)$\sim$(12) are solved by the fully implicit scheme, and Eqs.(8) and (12) are obtained by taking $\bar{u}_{k+2}^{n+1}-\bar{u}_{k+1}^{n+1}$ and $\tilde{u}_{k-2}^{n+1}-\tilde{u}_{k-1}^{n+1}$ of the fully implicit scheme as $u_{k+2}^n-u_{k+1}^n$ and $u_{k-2}^n-u_{k-1}^n$ respectively.

Using Eqs.(13), (4)$\sim$(12) can be rewritten as

$$\Delta_\tau u_j^{n+1}=\delta^2 u_j^{n+1},\quad j=1,2,\cdots,k-2,k+2,\cdots,J-1, \tag{14}$$

$$\Delta_\tau u_{k-1}^{n+1}=\delta^2 u_{k-1}^{n+1}+r_{k-1}^{n+1}, \tag{15}$$

$$\Delta_\tau u_k^{n+1}=\delta^2 u_k^{n+1}+r_k^{n+1}, \tag{16}$$

$$\Delta_\tau u_{k+1}^{n+1}=\delta^2 u_{k+1}^{n+1}+r_{k+1}^{n+1}, \tag{17}$$