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ON THE CONVERGENCE OF THE BRENT METHOD*[1)]

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Abstract

In this paper, we establish the semi-local convergence theorem of the Brent method with regional estimation. By an in-depth investigation in to the algorithm structure of the method, we convert the Brent method into an approximate Newton method with a special error term. Bsaed on such equivalent variation, under a similar condition of the Newton-Kantorovich theorem of the Newton method, we establish a semi-local convergence theorem of the Brent method. This theorem provides a sufficient theoretical basis for initial choices of the Brent method.

1. Introduction

It is well known that the Brent method for solving systems of nonlinear equations

$$F(x)=0, \qquad F: D\subset R^n \longrightarrow R^n \tag{1.1}$$

is to solve the following system:

$$\begin{aligned}
&J(x^{(k)},h^{(k)})(x^{(k+1)}-x^{(k)})+F(x^{(k)})=0,\\
&J(x^{(k)},h^{(k)})e_i=[F(x^{(k)}+h_i^{(k)}e_i)-F(x^{(k)})]/h_i^{(k)}, \quad h_i^{(k)}\neq 0,\\
&h^{(k)T}=[h_1^{(k)},\cdots,h_n^{(k)}], \quad e_i^T=[0,\cdots,0,1,0,\cdots,0],\\
&\qquad\qquad i=1,2,\cdots,n; \quad k=0,1,2,\cdots,
\end{aligned} \tag{1.2}$$

by making use of the orthogonal triangular factorization. Suppose that we have an approximation $x^{(k)}$ to x^*, a solution of (1.1). Then the k-th iterative procedure can be described as follows [1]:

Step 1. Let $y_1^{(k)}=x^{(k)}, Q_1^{(k)}=Q_{n+1}^{(k-1)}$(or $Q_1^{(k)}=I$).

Step 2. Compute the vector

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$$\bar{a}_j^{(k)} = \frac{1}{h_k}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ f_j(y_j^{(k)} + h_k Q_j^{(k)} e_j) - f_j(y_j^{(k)}) \\ \vdots \\ f_j(y_j^{(k)} + h_k Q_j^{(k)} e_n) - f_j(y_j^{(k)}) \end{bmatrix}, \tag{1.3}$$

where $h_k \neq 0$ is the difference step corresponding to the index k (we will discuss the choices of h_k in Section 4). Construct an orthogonal matrix (usually by the Household transformation)

$$U_j^{(k)} = \begin{bmatrix} I_{j-1} & 0 \\ 0 & \bar{U}_j^{(k)} \end{bmatrix} \tag{1.4}$$

such that

$$(\bar{a}_j^{(k)})^T U_j^{(k)} = \sigma_j^{(k)} e_j^T, \quad \sigma_j^{(k)} = \pm\|\bar{a}_j^{(k)}\|. \tag{1.5}$$

Let

$$Q_{j+1}^{(k)} = Q_j^{(k)} U_j^{(k)}. \tag{1.6}$$

Step 3. Compute

$$y_{j+1}^{(k)} = y_j^{(k)} - \frac{1}{\sigma_j^{(k)}} f_j(y_j^{(k)}) Q_{j+1}^{(k)} e_j. \tag{1.7}$$

Step 4. If $j < n$, let $j := j + 1$, go to Step 2; otherwise, let $x^{(k+1)} = y_{n+1}^{(k)}$.

Brent[1] applied the Shamanskii technique at each iteration to improve the efficiency of the algorithm. However, it is not essential to the discussion of this paper.

Brent[1] proved that the above algorithm converges locally to x^* with a quadratic convergence order. Since for each iteration the number of function evaluations of the algorithm is $\frac{n(n+3)}{2}$, which is nearly half of what is needed by the Newton method, and since it is also of satisfactory numerical stability, the Brent method has long been regarded as a most effective numerical method for solving nonlinear systems.

The main purpose of this paper is to establish the semi-local convergence theorem of the Brent method with regional estimations. Because of the complexity of the algorithm structure, the classical Kantorovich method can not be applied to this method. By investigation into the algorithm structure of the method, we convert the Brent method into an approximate Newton method with a special error term. Based on such equivalent variation, under a similar condition of the Newton-Kantorovich theorem of the Newton method, for the Brent method we establish an existence-convergence theorem (semilocal convergence theorem), which provides a sufficient theortical basis for initial choices of the Brent method but has not been proved for nearly twenty years.