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REGULARITY OF BIRKHOFF INTERPOLATION*[1)]

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Abstract

A comparison theorem concerning the regularity of Birkhoff interpolation is given. As an application of this theorem the regularity of $(0, 1, \cdots, p-1, p+1, \cdots, M-1, q)$ interpolation $(0 < p < M \le q)$ is characterized.

1. Introduction

The following definitions and notations are taken from [1, pp. 2–3].

Let $G = \{g_0, g_1, \cdots, g_N\}$ be a system of linearly independent, m times continuously differentiable functions on $[-1, 1]$. A matrix

$$E = [e_{ik};\quad i = 1, 2, \cdots, n,\quad k = 0, 1, \cdots, m],\quad n \ge 1,\quad m \ge 0 \tag{1.1}$$

is called an interpolation matrix if its elements e_{ik} are 0 or 1 and if the number of 1's in E is equal to $N+1$, $|E| = \sum e_{ik} = N+1$. Let X denote a set of knots

$$1 \ge x_1 > x_2 > \cdots > x_n \ge -1. \tag{1.2}$$

A Birkhoff interpolation problem E, X (with respect to G) is, given a set of data c_{ik} (defined for $e_{ik} = 1$) to determine a polynomial $P = \sum_{j=0}^{N} a_j g_j$ (if any) such that

$$P^{(k)}(x_i) = c_{ik},\quad e_{ik} = 1,\quad e_{ik} \in E. \tag{1.3}$$

The pair E, X is called regular if the system of equations (1.3) has a unique solution for each given set of c_{ik}; otherwise the pair E, X is singular. The matrix E is called order regular if the pair E, X is regular for any ordered set of knots X. Since the system (1.3) consists of $N+1$ linear equations with $N+1$ unknowns a_j, a pair E, X is regular if and only if the determinant of the system

$$D(E, X) := D(E, X; g_0, \cdots, g_N) = \det\,[g_0^{(k)}(x_i), \cdots, g_N^{(k)}(x_i);\quad e_{ik} = 1,\quad e_{ik} \in E] \tag{1.4}$$

is nonzero; or equivalently, a pair E, X is singular if and only if some nontrivial polynomial P $\in$span G is annihilated by E, X, i.e., P satisfies the homogeneous equations

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$P^{(k)}(x_i) = 0$ for $e_{ik} = 1$. We order the pair E, X in (1.4) lexicographically [1, p.3]. By $A(E, X)$ we denote the $(N+1) \times (N+1)$ matrix that appears in (1.4).

A function $P_{ik} = \sum_{j=0}^{N} a_j g_j$ with $e_{ik} = 1$ and $e_{ik} \in E$ is said to be a fundamental function for the pair E, X if

$$P_{ik}^{(\mu)}(x_\nu) = \delta_{i\nu}\delta_{k\mu}, \quad e_{\nu\mu} = 1, \ e_{\nu\mu} \in E. \tag{1.5}$$

Clearly the determinant (1.4) is often very complicated; it is difficult to claim whether or not $D(E, X)$ vanishes. Thus simplification of $D(E, X)$ is of important interest.

One of the objects of this paper is to establish a comparison theorem, which makes it possible to decrease the order of $D(E, X)$ and to simplify $D(E, X)$ (Section 2). Then, in Section 3, we apply this theorem to $(0, 1, \cdots, p-1, p+1, \cdots, M-1, q)$ interpolation $(0 < p < M \leq q \leq N-n+1)$. (Here we agree that such a interpolation is $(0, 1, \cdots, M-2, q)$ interpolation when $p = M-1$.) That is the problem E, X, where E is the $n \times (N+1)$ matrix with

$$e_{ik} = \begin{cases} 1, & i = 1, 2, \cdots, n, \quad k = 0, 1, \cdots, p-1, p+1, \cdots, M-1, q, \\ 0, & \text{otherwise}. \end{cases} \tag{1.6}$$

In what follows we restrict ourselves to the case when span $G = \mathbf{P}_N$, the set of algebraic polynomials of degree at most N. In this case we can assume that $m \leq N$, and by adding zero columns if necessary, we can make $m = N$. Such a matrix we shall call normal.

In the following we have to apply a theorem several times proved by Atkinson and Sharma [1, Theorem 1.5, p. 10]. For the sake of convenience we shall state it here. To this end we need some further definitions from [1, pp. 7-9].

For normal matrices the condition

$$\sum_{k=0}^{s} \sum_{i=1}^{n} e_{ik} \geq s+1, \quad s = 0, 1, \cdots, N \tag{1.7}$$

is called the Pólya condition. A sequence of 1's of the ith row of E is supported if that (i, k) is the position of the first 1 of the sequence implies that there exist two 1's: $e_{i_1,k_1} = e_{i_2,k_2} = 1$ with $i_1 < i < i_2$, $k_1 < k$, and $k_2 < k$. Then we have

Theorem A. *A normal interpolation matrix is order regular for algebraic interpolation if it satisfies the Pólya condition and contains no odd supported sequences.*

2. A Comparison Theorem

Let E, E_1, and E_2 be $n \times (N+1)$ matrices, not necessarily normal, the elements in which take 1 or 0. We write $E = E_1 + E_2$ if it stands for the ordinary addition of matrices. The main result in this section is the following theorem, a special case of which can be found in [1, Theorem 8.1, p. 101].