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MONOTONE ITERATION FOR ELLIPTIC PDEs WITH DISCONTINUOUS NONLINEAR TERMS*

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Abstract *In this paper, we use monotone iterative techniques to show the existence of maximal or minimal solutions of some elliptic PDEs with nonlinear discontinuous terms. As the numerical analysis of this PDEs is concerned, we prove the convergence of discrete extremal solutions.*
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1 Introduction

It is well known that the general monotone iterative techniques coupled with the upper and lower solutions offer effective and flexible mechanisms for proving theoretically the existence of the extremal(maximal/minimal) solutions for a variety of discontinuous nonlinear differential equation ([4,9,15,10]). In this paper, we first use monotone iteration to study the existence of extremal solutions of a type of discontinuous nonlinear partial differential equations. Then we study a problem deriving from the numerical computation of the extremal solutions. Suppose that we discretize the original PDEs by finite elements method and obtain a corresponding discrete system. Since the discrete systems have the same properties as the original one, we can obtain the existence of discrete extremal solutions. Now there exists a critical important problem: Does the discrete extremal solutions sequence converge to the extremal solutions of the original system. This convergence is difficult to be proven since the discrete extremal solutions sequences are no longer monotone and we could not use monotone iterative techniques again. To overcome this difficulty, we analyze the properties of the domain and use some embedding theorems of Sobolev spaces. The proof of this convergence offers a theoretical foundation for the numerical computation of the solution of the discontinuous nonlinear system.

We organize the rest of this paper as follows. In Section 2, we will give some preliminaries and

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assumptions about our system to be solved. In Section 3, we use monotone iterative techniques to show the existence of the extremal solutions of the system introduced in Section 2. Finally in section 4, we discretize the system by finite elements methods and prove the convergence of the discrete extremal solution sequences.

2 Elliptic PDEs with Discontinuous Nonlinear Terms

This section is dedicated to the presentation of elliptic PDEs with discontinuous nonlinear terms and their weak extremal solutions.

We begin with some notations and definitions. Let N be a positive integer and $\Omega \subset \mathbf{R}^N$ a bounded domain with the Lipschitz continuous boundary $\partial\Omega$. As usual, we denote by $C(\Omega), L_\infty(\Omega)$ the space containing continuous functions and essential bounded functions respectively. We let $L_2(\Omega)$ be the square-integrable function space with norm $\|\cdot\|_0$ defined for $v \in L_2(\Omega)$ by

$$\|v\|_0 := \left(\int_\Omega |v|^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

For all integer $t \geq 1$, we let $H^t(\Omega)$ be the standard Sobolev space with norm $\|\cdot\|_t$ defined for $v \in H^1(\Omega)$ by

$$\|v\|_t := \sum_{|\alpha|\leq t} \|D^\alpha v\|_0.$$

And as usual, we let the space

$$H_0^t(\Omega) := \{v \in H^t(\Omega) : \exists\ v_n \in H^t(\Omega),\ \mathrm{supp} v_n \subset \Omega,\ \text{such that } \lim_{n\to\infty} \|v_n - v\|_t = 0\}.$$

Now let $A(x) = (a_{ij}(x)), x \in \Omega, i, j \in \{1, \cdots, N\}$ be a $N \times N$ symmetric matrix whose entries a_{ij} are measurable functions defined on Ω. We furthermore assume that the matrix A satisfies the following two assumptions.

(A1) There exist a constant $c_0 \geq 0$ and a positive function $k_1 \in L_2(\Omega)$ such that the inequality

$$|\sum_{j=1}^N a_{ij}(x)\xi_j| \leq k_1(x) + c_0|\xi|$$

holds for almost all(a. a.) $x \in \Omega$, all $\xi = (\xi_1, \xi_2, \cdots, \xi_N) \in \mathbf{R}^N$ and for all $i \in \{1, \cdots, N\}$.

(A2) The matrix A is elliptic. That is, there exists a constant $c_1 > 0$ such that the inequality

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq c_1|\xi|^2$$

holds for a.a. $x \in \Omega$.

We suppose moreover that there exists a function c which is defined also on Ω and satisfies

(A3) $c \in L_\infty(\Omega)$ and $c(x) \geq 0$, for a.a. $x \in \Omega$.

To introduce the elliptic PDEs with discontinuous nonlinear terms, we also introduce two functions f and p which satisfy the following assumptions respectively.