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BOUNDARY ELEMENT APPROXIMATION OF THE SEMI-DISCRETE PARABOLIC VARIATIONAL INEQUALITIES OF THE SECOND KIND *

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Abstract *The boundary element approximation of the parabolic variational inequalities of the second kind is discussed. First, the parabolic variational inequalities of the second kind can be reduced to an elliptic variational inequality by using semi-discretization and implicit method in time; then the existence and uniqueness for the solution of nonlinear non-differentiable mixed variational inequality is discussed. Its corresponding mixed boundary variational inequality and the existence and uniqueness of its solution are yielded. This provides the theoretical basis for using boundary element method to solve the mixed variational inequality.*

Key words *parabolic variational inequalities; mixed boundary variational inequality.*
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1 Introduction

We consider the parabolic variational inequalities of the second kind with non-differentiable terms. Many works[1,6,7,9,10] related to this kind of parabolic variational inequalities have been done, for example, the combination of semi-discretization in time and finite element method [7]; generalized truncation method[1], etc. All these methods have the shortcoming of huge computation, especially for a long time problem. It's oppressive for the method of one finite element computation at each time step. In this paper. Following the frame of reference [4], we still use semi-discrete in time, then apply boundary element method to its equivalent variational equation. Then the original problem can be transferred into a mixed boundary variational inequality. By this way, we reach the aim to decrease the dimension of discussed problem and simplify the computation. The rest of the paper is structured as follows. In Section 2 we present the parabolic variational inequalities of the second kind and the corresponding equivalent unilateral boundary problem. In Section 3, the boundary integral equation by introducing homogeneous Helmholtz

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equation is established. In section 4, using boundary integral equation, we transferred the mixed variational inequality defined in spatial domain to the mixed boundary variational inequality integral on boundary and proved the existence and uniqueness of the solution of its corresponding boundary variational inequality. Finally, the numerical example is given.

2 The Parabolic Variational Inequality

Let Ω be a bounded open domain in R^2 with a smooth boundary Γ. Time interval is $[0,T]$, where $0 < T < \infty$. For $u, v \in V = H^1(\Omega)$, we define

The bilinear form $\tilde{a}\,(u,v) = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x + \int_\Omega uv \mathrm{d}x$.

The functional $j(v) = \int_\Gamma c\,|v|\,\mathrm{d}s$, where $c > 0$ is a given constant.

The linear form $L(v) = \int_\Omega \tilde{f} v \mathrm{d}x$.

$$\tilde{V} = \{\text{Trace of } v \text{ on } \Gamma \in V\} = \overset{*}{H}{}^{1/2}(\Gamma),$$
$$\overset{*}{H}{}^{-1/2}(\Gamma) = \left\{\mu \in H^{-1/2}(\Gamma), \textstyle\int_\Gamma \mu \mathrm{d}s = 0\right\}.$$

Here $H^m(\Gamma)$ and $H^\alpha(\Gamma)$ denote the Sobolev spaces with the order of integer and fraction respectively, and suppose $f \in H^1(\Gamma)$. It is easy to see that $\tilde{V}$ is a closed linear subspace of $H^1(\Gamma)$.

Consider the following parabolic variational inequalities of the second kind [1,6,7]

$$\begin{cases} \text{Find } u : [0,T] \to V, \text{such that,} & \\ (\dfrac{\partial u}{\partial t}, v-u) + \tilde{a}(u, v-u) + j(v) - j(u) \geq (\tilde{f}, v-u), & \forall v \in V, \text{ a. e. } t > 0, \\ u(x,0) = u^0(x), & \forall x \in \Omega \end{cases} \tag{2.1}$$

where $\tilde{f}, \tilde{f}_t \in C([0,T] \times \bar{\Omega}), u^0 \in V \cap W^2_\infty(\Omega)$.

Remark From references [1, 7] for any given $T > 0$, the variational inequality (2.1) has a unique solution. Moreover, the map $u(\cdot,t) \to V$ is continuous, $u_t \in L^2([0,T],V)$,and

$$\sup_{0 \leq t \leq T} \Big(\|u(\cdot,t)\|_{H^2(\Omega)} + \|u_t(\cdot,t)\|_{L^\infty(\Omega)} \Big) < \infty.$$

We take semi-discrete approximation and implicit method in time for (2.1). Let time step be $\Delta t = \dfrac{T}{N}$, denote $u^n = u(x,t^n)$, where $t^n = n\Delta t, (n = 0,1,\dots N)$. Let u^n be the approximate solution at time t^n, we compute the approximate solution u^{n+1} at time t^{n+1}

$$(\frac{u^{n+1} - u^n}{\Delta t}, v - u^{n+1}) + \tilde{a}(u^{n+1}, v - u^{n+1}) + j(v) - j(u^{n+1}) \geq (\tilde{f}^{n+1}, v - u^{n+1}). \tag{2.2}$$

According to [10], the stability of (2.2) is unconditional. Rewrite (2.2) as

$$\tilde{a}(u^{n+1}, v - u^{n+1}) + \frac{1}{\Delta t}(u^{n+1}, v - u^{n+1}) + j(v) - j(u^{n+1}) \geq (\tilde{f}^{n+1} + \frac{1}{\Delta t}u^n, v - u^{n+1}). \tag{2.3}$$