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THE MONOTONICITY OF CONVERGENCE RATE FOR MGS METHODS*

Wang Yongjun(王永俊)

Abstract *In this paper we prove that the asymptotic rate of convergence of the modified Gauss-Seidel method of a non-singular M-matrix is a monotonic function for precondition parameters* $0 \leqslant \alpha_i \leqslant \frac{1}{2}$, $(i = 1, 2, \cdots, n-1)$.

Key words *Gauss-seidel method, convergence rate, monotonicity.*

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1 Introduction

Let A be an $n \times n$ matrix with all diagonal entries 1, $-L$ and $-U$ be strictly lower and strictly upper triangular part of A, respectively. Then the Gauss-Seidel splitting of A has the form that $A = (I - L) - U$, where I is the identity matrix of order n. For the convenience of statement, we take the notations as follows:

$$V = \begin{bmatrix} 0 & -a_{12} & 0 & \cdots & 0 \\ 0 & 0 & -a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{n-1})$, $D_\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, 1)$, $S_\alpha = D_\alpha V$, $P_\alpha = I + S_\alpha$, $A_\alpha = P_\alpha A$. Briefly, we denote D_c, S_c, P_c, A_c, etc. for the case α_i $(\forall i)$ all c, respectively.

Consider Gauss-Seidel splitting $A_\alpha \doteq E_\alpha - F_\alpha$. Observably, A_0 is the case of standard Gauss-Seidel splitting of A. Gunawardena et al [1] studied firstly the convergence for A_1 (the Modified Gauss-Seidel method). And then Kohno et al [2] extended to the general case for $0 \leqslant \alpha \leqslant 1$.

When A is a non-singular M-matrix, the iterative matrix $T_\alpha = E_\alpha^{-1} F_\alpha$ of Gauss-Seidel splitting for A_α is non-negative, and has the spectral radius $\rho_\alpha = \rho(T_\alpha) < 1$. Gunawardena's works [1] show that $\rho_1 \leqslant \rho_0$, and Li's works [3] show that $\rho_\alpha \leqslant \rho_0$ for $0 \leqslant \alpha \leqslant 1$. In [4], Li shows that $\rho_\alpha \geqslant \rho_\beta$ for $0 \leqslant \alpha \leqslant \beta \leqslant \varepsilon$, where ε is some vector only relative to matrix A when A is a diagonally dominant non-singular M-matrix, and conjectures that the above statement would be true for $0 \leqslant \alpha \leqslant \beta \leqslant 1$. That is, ρ_α would be a monotonic function when $0 \leqslant \alpha \leqslant 1$.

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In this paper, we show Li's conjecture true for any non-singular *M-matrix* when $0 \leqslant \alpha \leqslant \xi$, (where $\xi \geqslant \dfrac{1}{2}$ and only relative to A) without the assumption that A is diagonally dominant.

2 Some Facts and Lemmas

Throughout the rest of this paper, we always suppose that A is a non-singular *M-matrix*, $0 \leqslant \alpha \leqslant \beta \leqslant 1$, and $M_\varepsilon = P_\varepsilon^{-1}E_\varepsilon$, $N_\varepsilon = P_\varepsilon^{-1}F_\varepsilon$, x_ε is a non-negative eigenvector belonging to ρ_ε of T_ε (where $\varepsilon = \alpha, \beta$). By simple computing, following facts can be obtained :

$$N_\varepsilon = (U - S_\varepsilon) + S_\varepsilon^2 \left(I - S_\varepsilon^2\right)^{-1}(I - S_\varepsilon), \tag{2.1}$$

$$M_\varepsilon^{-1}N_\varepsilon = E_\varepsilon^{-1}F_\varepsilon = T_\varepsilon \geqslant 0, \tag{2.2}$$

$$F_\varepsilon \geqslant 0. \tag{2.3}$$

Some lemmas without proof are stated as follows, which can be easily followed from [4][5]:

Lemma 2.1 If $\rho_\varepsilon > 0$, $Ax_\varepsilon = \dfrac{1-\rho_\varepsilon}{\rho_\varepsilon}N_\varepsilon x_\varepsilon$, and $A_\varepsilon x_\varepsilon = \dfrac{1-\rho_\varepsilon}{\rho_\varepsilon}F_\varepsilon x_\varepsilon \geqslant 0$.

Lemma 2.2 $T_\alpha A^{-1} \geqslant T_\beta A^{-1}$.

Lemma 2.3 $A_\varepsilon = P_\varepsilon A$ is a non-singular M-matrix.

3 Results

Now, we show our main theorem.

Theorem 3.1 Let A be a non-singular M-matrix, $0 \leqslant \alpha \leqslant \beta \leqslant \xi$, where $\xi_i = \dfrac{1}{1+\sqrt{1-a_{i,i+1}a_{i+1,i}}}$ $(0 \leqslant i < n)$. Then $\rho_\alpha \geqslant \rho_\beta$.

Proof When $\rho_\beta = 0$, $\rho_\alpha \geqslant 0 = \rho_\beta$, it obvious.

Now suppose that $0 < \rho_\beta < 1$.

Let $q_i = \dfrac{1}{1-\beta_i a_{i,i+1}a_{i+1,i}}$, $Q_\beta = \operatorname{diag}(q_1, \cdots, q_{n-1}, 1)$, $s_i = -\beta_i a_{i,i+1}$, $(0 \leqslant i < n)$. Because A is a non-singular M-matrix, $1 > a_{i,i+1}a_{i+1,i}$. So, $\xi_i < 1 \leqslant q_i$, $(0 \leqslant i < n)$. Then

$$\begin{aligned}(Q_\beta A_\beta - (I - S_\beta))_{i,i+1} &= (Q_\beta A_\beta + S_\beta)_{i,i+1} = q_i(1-\beta_i)a_{i,i+1} + s_i \\ &= \frac{(1-\beta_i)a_{i,i+1}}{1-\beta_i a_{i,i+1}a_{i+1,i}} - \beta_i a_{i,i+1} = q_i a_{i,i+1}\cdot(1 - 2\beta_i + \beta_i^2 a_{i,i+1}a_{i+1,i}).\end{aligned}$$

If $a_{i,i+1}a_{i+1,i} = 0$, then $1 - 2\beta_i \geqslant 1 - 2\xi_i = 0$. So, $(I - S_\beta)_{i,i+1} \geqslant (Q_\beta A_\beta)_{i,i+1}$ because of $a_{i,i+1} \leqslant 0$. While $a_{i,i+1}a_{i+1,i} > 0$, we have that

$$\begin{aligned}1 - 2\beta_i + \beta_i^2 a_{i,i+1}a_{i+1,i} &= \left(\frac{1}{1-\sqrt{1-a_{i,i+1}a_{i+1,i}}} - \beta_i\right)\cdot\left(\frac{1}{1+\sqrt{1-a_{i,i+1}a_{i+1,i}}} - \beta_i\right) \\ &= \left(\frac{1}{1-\sqrt{1-a_{i,i+1}a_{i+1,i}}} - \beta_i\right)\cdot(\xi_i - \beta_i).\end{aligned}$$