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Maximum Modulus of Polynomials

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Abstract. Let

$$P(z)=\sum_{j=0}^{n} a_j z^j$$

be a polynomial of degree n and let $M(P,r)=\max_{|z|=r}|P(z)|$. If $P(z)\neq 0$ in $|z|<1$, then

$$M(P,r)\geq \left(\frac{1+r}{1+\rho}\right)^n M(P,\rho).$$

The result is best possible. In this paper we shall present a refinement of this result and some other related results.

Key Words: Maximum modulus, growth of polynomial, derivative.

AMS Subject Classifications: 30A10, 30C10, 30C15

1 Introduction and statement of results

Let

$$P(z)=\sum_{j=0}^{n} a_j z^j$$

be a polynomial of degree n, let

$$M(P,r)=\max_{|z|=r}|P(z)| \quad \text{and} \quad m(P,1)=\min_{|z|=1}|P(z)|,$$

then concerning the size of $M(P,r)$ the following results are well known.

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Theorem 1.1 (Bernstein [3]). *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n, then*

$$M(P,R)\le R^nM(P,1) \quad \text{for } R\ge 1 \tag{1.1}$$

with equality only for $P(z)=\lambda z^n$.

Theorem 1.2 (Zarantauello and Verga [6]). *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n, then*

$$M(P,r)\ge r^nM(P,1) \quad \text{for } r\le 1 \tag{1.2}$$

with equality only for $P(z)=\lambda z^n$.

For polynomials not vanishing in $|z|<1$, Rivilin [5] proved:

Theorem 1.3. *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n, $P(z)\neq 0$ in $|z|<1$, then*

$$M(P,r)\ge\left(\frac{1+r}{2}\right)^nM(P,1) \quad \text{for } r\le 1. \tag{1.3}$$

The result is best possible with equality only for the polynomial

$$P(z)=\left(\frac{\lambda+\mu z}{2}\right)^n, \quad |\lambda|=|\mu|.$$

Govil [2] has proved the following generalization of Theorem 1.3.

Theorem 1.4. *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n having no zero in $|z|<1$, then for $0\le r\le\rho\le 1$,*

$$M(P,r)\ge\left(\frac{1+r}{1+\rho}\right)^nM(P,\rho). \tag{1.4}$$

The result is best possible and equality holds for the polynomial

$$P(z)=\left(\frac{1+z}{1+\rho}\right)^n.$$

He has shown that the bound can be improved if $P'(0)=0$ and proved:

Theorem 1.5. *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n, having no zero in $|z|<1$, $P'(0)=0$ then for $0\le r\le\rho\le 1$,*

$$M(P,r)\ge\left(\frac{1+r}{1+\rho}\right)^n\left\{\frac{1}{1-\frac{(1-\rho)(\rho-r)n}{4}\left(\frac{1+r}{1+\rho}\right)^{n-1}}\right\}M(P,\rho). \tag{1.5}$$

In this paper, we shall present the following refinements of Theorems 1.4 and 1.5. Here we prove: