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Regularity Criteria for the Navier-Stokes Equations Containing Two Velocity Components

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Abstract. Based on the critical sobolev inequalities in the Besov spaces with the logarithmic form, the regularity criteria in terms of two velocity components for the 3D incompressible Navier-Stokes equations is improved.

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1 Introduction

In this paper, we consider the following incompressible Navier-Stokes equations in $\mathbb{R}^3 \times (0,T)$

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \Delta u + \nabla p = 0 \\ \quad \nabla\cdot u = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

Where $u=u(x,t)$ is the velocity field, $p(x,t)$ is the scalar pressure and $u_0(x)$ with $\nabla\cdot u_0=0$ in the sense of distribution is the initial velocity field.

It is well known that for $u_0(x)\in L^2(\mathbb{R}^3)$, (1.1) exist at least one weak solution that is called Leray-Hopf weak solution. Nevertheless, the fundamental problem of the uniqueness and regularity of such solutions is still open.

However, the solution regularity can been derived when certain growth conditions are satisfied. This is known as a regularity criterion problem introduced in the celebrated work of Serrin [1], and can be described as follows:

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A weak solution u is regular if the growth condition

$$u\in L^p(0,T;L^q(\mathbb{R}^3)); \quad \frac{2}{p}+\frac{3}{q}=1, \quad 3<q\le\infty, \tag{1.2}$$

holds true.

There is a large literature on improvement of the condition (1.2). It may be superfluous to recall all results. To go directly to the main points of the present paper, we only review some known results which are closely related to our main result. Beirã da Veiga [2] improved the condition (1.2) in terms of two velocity components

$$\tilde{u}\in L^p(0,T;L^q(\mathbb{R}^3)); \quad \frac{2}{p}+\frac{3}{q}=1, \quad 3<q\le\infty, \tag{1.3}$$

where $\tilde{u}=(u_1,u_2,0)$ is the horizontal velocity.

Very recently, Zhang [3] extended the condition (1.3) into BMO space in the marginal case $q=\infty$

$$\tilde{u}\in L^2(0,T;BMO). \tag{1.4}$$

Another interesting contribution of this problem is due to Beirão da Veiga [4] on the regularity criterion with respect to the velocity gradient condition

$$\nabla u\in L^p(0,T;L^q(\mathbb{R}^3)); \quad \frac{2}{p}+\frac{3}{q}=2, \quad \frac{3}{2}<q\le\infty. \tag{1.5}$$

Recently, based on the Littelewood-Paley decomposition to the equations(1.1), Dong and Zhang [5] extended the regularity criterion via two components of velocity field in homogeneous Besov space

$$\nabla_h\tilde{u}\in L^2(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)); \quad \nabla_h\tilde{u}=(\partial_1\tilde{u},\partial_2\tilde{u}). \tag{1.6}$$

Penel and Pokorny [6] obtained an improved regularity result in L^p spaces

$$\partial_1u_1,\partial_2u_2\in L^p(0,T;L^q(\mathbb{R}^3)); \quad \frac{2}{p}+\frac{3}{q}=2, \quad \frac{3}{2}<q\le\infty. \tag{1.7}$$

Dong and Chen [7] improved the condition (1.7) in Lorentz space, Morrey space and multiplier space. Actually, the weak solution remains regular if the single velocity component satisfies some conditions (see [8, 9, 16]).

The aim of this present paper is to extend the regularity criterion (1.6) and (1.7) in homogeneous Besov space in the marginal case. More precisely, we will prove the following result.

Theorem 1.1. *Suppose $u_0\in H^3(\mathbb{R}^3)$ and $\nabla\cdot u_0=0$ in the sense of distributions. Assume that u is a Leray-Hopf weak solutions of (1.1) on $(0,T)$. If u satisfies the following condition*

$$\int_0^T \frac{\|(\partial_1u_1,\partial_2u_2)\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1+log(1+\|(\partial_1u_1,\partial_2u_2)\|_{\dot{B}^0_{\infty,\infty}})}}dt<\infty, \tag{1.8}$$

then the weak solution is regular on $(0,T]$.

A very interesting consequence of (1.8) is the condition

$$\int_0^T \|(\partial_1 u_1, \partial_2 u_2)\|_{\dot{B}^0_{\infty,\infty}} dt < \infty, \tag{1.9}$$

which is a refined improvement of the famous BKM criterion [11] to Navier-Stokes equations, and the same time is also an improvement of the condition (1.6).

Remark 1.1. Throughout the paper, C stands for a constant, and changes from line to line; $\|.\|_p$ denotes the norm of the Lebesgue space L^p.

2 Preliminaries

We first introduce the Littlewood-Paley decomposition and the definition of Besov spaces, one may check [12, 13] for more details.

For $f \in \mathcal{S}$, the Schwartz class of rapidly decreasing functions, define the Fourier transform

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x) dx.$$

Choose two nonnegative radial functions $\chi, \varphi \in \mathcal{S}(\mathbb{R}^3)$ supported respectively in $\mathcal{B} = \{\xi \in \mathbb{R}^3, |\xi| \le \frac{4}{3}\}$ and $\mathcal{C} = \{\xi \in \mathbb{R}^3, \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$ such that

$$\chi(\xi) + \sum_{j \ge 0} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^3.$$

We define the pseudo-differential operators

$$\triangle_{-1} f = \mathcal{F}^{-1}\big(\chi(\xi)\hat{f}(\xi)\big), \quad \triangle_j f = \mathcal{F}^{-1}\big(\varphi(2^{-j}\xi)\hat{f}(\xi)\big), \quad j \ge 0,$$

and set

$$\mathcal{S}_j f = \sum_{k=-1}^{j-1} \triangle_k f.$$

By telescoping the series, we thus have the following Littlewood-Paley decomposition

$$f = \sum_{j=-\infty}^{\infty} \triangle_j f, \quad f \in L^2(\mathbb{R}^3). \tag{2.1}$$

The homogeneous Besov space $\dot{B}^s_{p,q}$ is defined by the semi-norm

$$\dot{B}^s_{p,q} = \{ f \in \mathcal{S}'(\mathbb{R}^3); \ \|f\|_{\dot{B}^s_{p,q}} < \infty \},$$

where

$$\|f\|_{\dot{B}^s_{p,q}}=\begin{cases}(\sum\limits_{j=-\infty}^{\infty}2^{jqs}\|\triangle_j f(\cdot)\|_p^q)^{\frac{1}{q}}, & \text{if } q\in[1,\infty)\\ \sup\limits_{j\in Z}2^{js}\|\triangle_j f(\cdot)\|_p, & \text{if } q=\infty.\end{cases} \tag{2.2}$$

It is deserved to point out that $\dot{B}^s_{2,2}$ is the homogenous Sobolev space $\dot{H}$ and has the following imbedding

$$L^\infty(\mathbb{R}^3)\subset BMO\subset \dot{B}^0_{\infty,\infty}(\mathbb{R}^3),$$

where BMO is the space of the Bounded Mean Oscillation. BMO is the space defined as a set for $L^1_{loc}\mathbb{R}^3$ function f such that

$$\|f\|^2_{BMO}=\sup_{x\in\mathbb{R}^3}\sup_{x\in\mathcal{R}}\frac{1}{B(x,R)}\int_{B(x,R)}|f(y)-\tilde{f}_{B_R}(y)|dy<\infty,$$

where $\tilde{f}_{B_R}$ denote for the average of f over all ball $B_R(x)$ in $\mathbb{R}^3$.

3 Proof of Theorem 1.1

Before going to the proof, we recall the following two inequalities established in [14,15] respectively

$$\|f\cdot\nabla f\|_r\leq C\|f\|_r\|\nabla f\|_{BMo}, \tag{3.1}$$

for $f\in W^{1,r}$ with $\nabla\cdot f=0$ and

$$\|f\|_{BMO}\leq C(1+\|f\|_{\dot{B}^0_{\infty,\infty}}log^{\frac{1}{2}}(1+\|f\|_{H^{s-1}})), \tag{3.2}$$

for $f\in H^{s-1}$ with $s>\frac{n}{2}+1$.

Remark 3.1. The idea of this proof is based on the method from [16,17].

Taking the inner product of the $i-$th equation of (1.1) with $|u_i|^2u_i$, $(i=1,2,3)$ and integrating by parts, we can show that

$$\sum_{i=1}^{3}(\frac{1}{4}\frac{d}{dt}\|u_i\|_4^4+3\||u_i|\nabla u_i\|_2^2)\leq 3\sum_{i=1}^{3}\int_{\mathbb{R}^3}p|u_i|^2\partial_i u_i dx. \tag{3.3}$$

Here we have used the following equality, due to the free divergence condition

$$\sum_{i=1}^{3}\int_{\mathbb{R}^3}(u\cdot\nabla)u_i|u_i|^2u_i dx=0.$$

Applying the inequalities (3.1) and (3.2), we have

$$\begin{aligned}\int_{\mathbb{R}^3} p|u_i|^2\partial_i u_i dx &\leq C\|p\|_2\|\partial_i u_i\|_{BMO}\|u_i\|_4^2\\ &\leq C\|\partial_i u_i\|_{BMO}\|u_i\|_4^4\\ &\leq C(1+\|\partial_i u_i\|_{\dot{B}^0_{\infty,\infty}} log^{\frac{1}{2}}(1+\|\nabla^3 u\|_2))\|u\|_4^4\\ &\leq C(1+\frac{\|(\partial_1 u_1,\partial_2 u_2)\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1+log(1+\|(\partial_1 u_1,\partial_2 u_2)\|_{\dot{B}^0_{\infty,\infty}})}} log(1+\|\nabla^3 u\|_2))\|u\|_4^4. \quad (3.4)\end{aligned}$$

Here we have used the following relationship between p and u

$$\|p\|_p \leq C\|u\|_{2p}^2, \;\; p\in(1,+\infty).$$

Now, we take

$$y(t)=\sup_{t\in[T_*,T]}\|\nabla^3 u\|_2,$$

then due to (3.4), we have

$$\sup_{t\in[T_*,T]}\|u(t)\|_4^4\leq C_*(1+y(t))^{C\epsilon},$$

where ϵ is a small constant, such that

$$\int_{T_*}^T \frac{\|(\partial_1 u_1,\partial_2 u_2)\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1+log(1+\|(\partial_1 u_1,\partial_2 u_2)\|_{\dot{B}^0_{\infty,\infty}})}}dt<\epsilon,$$

where C_* is a positive constant depending on T_*.

Then, we do estimate for $\|\nabla u\|_2$.

Multiplying the first equation of (1.1) by $-\Delta u$, after integration by parts, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2+\|\Delta u\|_2^2 &= \int_{\mathbb{R}^3}(u\cdot\nabla u)u\cdot\Delta u dx\\ &\leq \|u\|_4\|\nabla u\|_4\|\Delta u\|_2\\ &\leq C\|u\|_4^8\|\nabla u\|_2^2+C\|\Delta u\|_2^2\\ &\leq C\|u\|_4^{16}+C\|\nabla u\|_2^4+C\|\Delta u\|_2^2, \quad (3.5)\end{aligned}$$

where we have used the inequality $\|\nabla u\|_4\leq\|\nabla u\|_2^{\frac{1}{4}}\|\Delta u\|_2^{\frac{3}{4}}$.

Integrating (3.5)on $[T_*,t]$, we have

$$\sup_{t\in[T_*,T]}\|\nabla u(t)\|_2^2\leq C(1+y(t))^{4C\epsilon}+C. \quad (3.6)$$

Applying ∇^3 to the first equation of (1.1), then taking L^2 inner product of the resulting equation with ∇^3, using integration by parts, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla^3 u\|_2^2+\|\nabla^4 u\|_2^2 &= -\int_{\mathbb{R}^3}\nabla^3[(u\cdot\nabla)u]\nabla^3 u dx \\
&= -\int_{\mathbb{R}^3}[\nabla^3(u\cdot\nabla u)-(u\cdot\nabla)\nabla^3 u]\nabla^3 u dx \\
&\le \|\nabla^3(u\cdot\nabla u)-(u\cdot\nabla)\nabla^3 u\|_2\|\nabla^3 u\|_2 \\
&\le C\|\nabla^3 u\|_4\|\nabla u\|_4\|\nabla^3 u\|_2 \\
&\le C\|\nabla^3 u\|_2^{\frac{5}{4}}\|\nabla^4 u\|_2^{\frac{3}{4}}\|\nabla u\|_2^{\frac{1}{4}}\|\nabla^2 u\|_2^{\frac{3}{4}} \\
&\le C\|\nabla^3 u\|_2^{\frac{5}{4}}\|\nabla^4 u\|_2\|\nabla u\|_2^{\frac{3}{4}} \\
&\le C\|\nabla^3 u\|_2^{\frac{1}{4}}\|\nabla^4 u\|_2^{\frac{5}{3}}\|\nabla u\|_2^{\frac{13}{12}} \\
&\le C\|\nabla^3 u\|_2^{\frac{3}{2}}\|\nabla u\|_2^{\frac{13}{2}}+C\|\nabla^4 u\|_2^2,
\end{aligned} \tag{3.7}$$

where we used the following Gagliardo-Nirenberg inequality:

$$\|\nabla^2 u\|_2\le C\|\nabla u\|_2^{\frac{2}{3}}\|\nabla^4 u\|_2^{\frac{1}{3}},$$

and

$$\|\nabla^3 u\|_2\le C\|\nabla u\|_2^{\frac{1}{3}}\|\nabla^4 u\|_2^{\frac{2}{3}}.$$

It should be clear to readers that applying Gronwall's inequality to (3.7) and taking (3.6) into account, we can obtain $y(t)\le C$. This completes the proof of Theorem 1.1.

References

[1] Serrin J. On the interior regularity of weak solutions of the Navier Stokes equations. Arch. Rational Mech. Anal., 9: 187-195, 1962.

[2] H. Beirão da Veiga. On the smoothness of a class of weak solutions to the Navier-Stokes equa- tions. J. Math. Fluid Mech., 2: 315-323, 2000.

[3] X.W. Zhang, W.L. Zhang and B.Q. Dong. Remarks on the regularity criteria of three-dimensional Navier-Stokes equations in margin case. J. Part. Diff. Eq., 24: 70- 82, 2011.

[4] H. Beirão da Veiga. A new regularity class for the Navier-Stokes equations in R^n. Chin. Ann. Math., 16: 407-412, 1995.

[5] B.Q. Dong and Z. Zhang. The BKM criterion for the 3D Navier-Stokes equations via two velocity components. Nonlinear Analysis: Real World Applications, 11: 2415-2421, 2010.

[6] P. Penel and M. Pokorny. Some new regularity criteria for the Navier-Stokes equations containing gradient of the velocity. Appl. Math., 49: 483-493, 2004.

[7] B.Q. Dong and Z.M. Cheng. On the regularity criteria of the 3D Navier-Stokes equations in critical spaces. Acta. Math. Sci., 31(B): 591-600, 2011.

[8] Y. Zhou. A new regularity criterion for the Navier-Stokes equations in terms of the gradient of one velocity component. Methods Appl. Anal., 9: 563-578, 2002.

[9] Y. Zhou and M. Pokorny. On the regularity of the solutions of the Navier-Stokes equations via one velocity component. Nonlinearity, 23: 1097-1107, 2010.

[10] I. Kukavica and M. Ziane. One component regularity for the Navier-Stokes equations. Nonlinearity, 19: 453-469, 2006.

[11] J.T. Beale, T. Kato and A. Majda. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. Commu. Math. Phys., 94: 61-66, 1984.

[12] J.-Y. Chemin. Perfect incompressible fluids. Oxford University Press, New York, 1998.

[13] E.M. Stein. Harmonic analysis: real-variable methods and orthogonality and Oscillatory integrals. Princeton University Press, New Jersey, 1993.

[14] H. Kozono, T. Ogawa and Y. Taniuchi. The critical sobolev inequalities in Besov spaces and regularity criterion to some semi-linear evolution equations. Math. Z., 242: 251-278, 2002.

[15] H. Kozono, T. Ogawa and Y. Taniuchi. Bilinear estimates in BMO and the Navier-Stokes equations. Math. Z., 235: 173-194, 2002.

[16] J.S. Fan, S. Jiang, N. Gen and Y. Zhou. Logrithmically improved regularity criteria for the Navier-Stokes and MHD equations. J. Math. Fluid Mech., DOI: 10.1007/s00021- 010-0039-5.

[17] Y. Zhou and J.S. Fan. Logarithmically improved regularity criteria for the 3D viscous MHD equations. Forum Mathematicum, 24: 691-708, 2012.