AN UPWIND DIFFERENCE SCHEME AND ITS CORRESPONDING BOUNDARY SCHEME*

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Abstract

An upwind difference scheme was given by the author in [5] for the numerical solution of steady-state problems. The present work studies this upwind scheme and its corresponding boundary scheme for the numerical solution of unsteady problems. For interior points the difference equations are approximations of the characteristic relations; for boundary points the difference equatons are approximations of the characteristic relations corresponding to the outgoing characteristics and the "non-reflecting" boundary conditions. Calculation of a Riemann problem in a finite computational region yields promising numerical results.

1. Introduction

Upwind difference schemes have always played important roles in the field of numerical solution of hyperbolic partial differential equations. We have, amongst the widely-used ones, the Courant, Isaacson, and Rees scheme [1], and the Godunov scheme, which is equivilent to an upwind scheme in a certain sense, see [2]. We also have [10], [3], [4], and many others. The author presented an upwind scheme in [5] for the numerical solution of steady-state problems. Its special feature is that the viscosity term concerned has effect in the unsteady process——it speeds up convergence; it has effect in the steady-state only in the shock region——it yields numerical shocks with at most one point of transition, but it does not influence the solution in the smooth region. Actually, the influence of vis-cosity on the smooth part of the solution depends directly on the boundary conditions. With suitable boundary conditions, the method under description consists of embedding a steady-state first order difference problem into an unsteady, second order (in space) difference problem. In [6] the author discussed the boundary scheme corresponding to the given upwind scheme and extended the boundary scheme to two-dimensional steady-state problems. This boundary scheme approximates the desired characteristic relations, it is in conservational form in the steady-state, and its implementation is especially convenient with implicit schemes.

The present work is an attempt to apply these schemes to the numerical solution of unsteady problems. A discussion of the upwind scheme and the corresponding boundary scheme is given in § 2. We shall see that for interior points, the difference equations are approximations of the characteristic relations. For boundary points,

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the difference equations are approximations of the characteristic relations corresponding to the outgoing characteristics, i. e. the desired characteristic relations, and the "non-reflecting" boundary conditions. When the region of solution is infinite, it is necessary to reduce it to a finite region for numerical solution. Then the "non-reflecting" boundary conditions have important significance. In § 3 the numerical solution of a Riemann problem in a finite computational region with the schemes discussed in § 2 is given. We shall see that the shock width is not large and that even when the rarefaction wave and the shock wave go out of the computational region, there are no apparent reflections from the boundaries.

2. An Upwind Scheme and Its Boundary Scheme

Consider the hyperbolic system

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}=0, \tag{1}$$

where U and F are p-dimensional vectors, $F=F(U)$. Let

$$A=\frac{\partial F}{\partial U},$$

then (1) can also be written as

$$\frac{\partial U}{\partial t}+A\frac{\partial U}{\partial x}=0.$$

Suppose $\lambda_1, \lambda_2, \cdots, \lambda_p$ are the eigenvalues of A, and $r_1, r_2, \cdots, r_p$ the corresponding right eigenvectors, then

$$A=RAR^{-1},\quad \Lambda=\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_p\end{pmatrix},$$

where R is the matrix $(r_1, r_2, \cdots, r_p)$. Let $L=R^{-1}$, then

$$L=\begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_p\end{pmatrix},$$

here l_i is the left eigenvector corresponding to λ_i. The characteristic normal form of (1) is

$$L\frac{\partial U}{\partial t}+\Lambda L\frac{\partial U}{\partial x}=0, \tag{2}$$

these are also called the characteristic relations.

The explicit form of the upwind scheme given in [5] is

$$\frac{U_j^{n+1}-U_j^n}{\Delta t}+\frac{F_{j+1}^n-F_{j-1}^n}{2\Delta x}=\frac{1}{2\Delta x}\left[\operatorname{sign} A_{j+\frac{1}{2}}^n\,(F_{j+1}^n-F_j^n)-\operatorname{sign} A_{j-\frac{1}{2}}^n(F_j^n-F_{j-1}^n)\right], \tag{3}$$

with $\operatorname{sign} A$ defined as follows

$$\operatorname{sign} A = R \cdot \operatorname{sign} \Lambda \cdot R^{-1}, \quad \operatorname{sign} \Lambda = \begin{pmatrix} \operatorname{sign}(\lambda_1) & & & \\ & \operatorname{sign}(\lambda_2) & & \\ & & \ddots & \\ & & & \operatorname{sign}(\lambda_p) \end{pmatrix}. \tag{4}$$

The above difference scheme can be approximated by

$$\frac{U_j^{n+1}-U_j^n}{\Delta t} + \frac{1}{2}(I+\operatorname{sign} A_{j-\frac{1}{2}}^n) A_{j-\frac{1}{2}}^n \frac{U_j^n - U_{j-1}^n}{\Delta x} + \frac{1}{2}(I-\operatorname{sign} A_{j+\frac{1}{2}}^n) A_{j+\frac{1}{2}}^n \frac{U_{j+1}^n - U_j^n}{\Delta x} = 0.$$

When the mesh is sufficiently small, we have $R_{j+\frac{1}{2}}^n \simeq R_{j-\frac{1}{2}}^n \simeq R$, $L_{j+\frac{1}{2}}^n \simeq L_{j-\frac{1}{2}}^n \simeq L$, for some R and L. Then

$$L\frac{U_j^{n+1}-U_j^n}{\Delta t} + \frac{1}{2}(I+\operatorname{sign} \Lambda_{j-\frac{1}{2}}^n) \Lambda_{j-\frac{1}{2}}^n L \frac{U_j^n - U_{j-1}^n}{\Delta x}$$
$$+ \frac{1}{2}(I-\operatorname{sign} \Lambda_{j+\frac{1}{2}}^n) \Lambda_{j+\frac{1}{2}}^n L \frac{U_{j+1}^n - U_j^n}{\Delta x} = 0.$$

Notice $\frac{1}{2}(I \pm \operatorname{sign} \Lambda_{j\mp\frac{1}{2}}^n) \Lambda_{j\mp\frac{1}{2}}^n$ is diagonal, so for $\lambda_{j-\frac{1}{2}}^n > 0$, $\lambda_{j+\frac{1}{2}}^n > 0$, the corresponding component equation is

$$l \cdot \frac{U_j^{n+1}-U_j^n}{\Delta t} + \lambda_{j-\frac{1}{2}}^n l \cdot \frac{U_j^n - U_{j-1}^n}{\Delta x} = 0, \tag{5a}$$

where the component subscript has been dropped. For $\lambda_{j-\frac{1}{2}}^n < 0$, $\lambda_{j+\frac{1}{2}}^n < 0$, we have

$$l \cdot \frac{U_j^{n+1}-U_j^n}{\Delta t} + \lambda_{j+\frac{1}{2}}^n l \cdot \frac{U_{j+1}^n - U_j^n}{\Delta x} = 0. \tag{5b}$$

For eigenvalues varying about zero, we consider each possible case separately. For $\lambda_{j-\frac{1}{2}}^n = \lambda_{j+\frac{1}{2}}^n = 0$, we get

$$l \cdot \frac{U_j^{n+1}-U_j^n}{\Delta t} = 0. \tag{5c}$$

For $\lambda_{j-\frac{1}{2}}^n < 0$, $\lambda_{j+\frac{1}{2}}^n > 0$, we also get (5c). For $\lambda_{j-\frac{1}{2}}^n > 0$, $\lambda_{j+\frac{1}{2}}^n < 0$, we obtain

$$l \cdot \frac{U_j^{n+1}-U_j^n}{\Delta t} + \left|\lambda_{j-\frac{1}{2}}^n\right| l \cdot \frac{U_j^n - U_{j-1}^n}{\Delta x} - \left|\lambda_{j+\frac{1}{2}}^n\right| l \cdot \frac{U_{j+1}^n - U_j^n}{\Delta x} = 0,$$

or

$$l \cdot \frac{U_j^{n+1}-U_j^n}{\Delta t} + O(\Delta x) = 0. \tag{5c'}$$

All other cases result in either (5c) or (5c′) in the same manner. In short, for eigenvalues >0, <0, or $\simeq 0$, the upwind scheme (3) is an approximation of the characteristic relations (2). We see also from (5) that for positive eigenvalues we have backward differences; and for negative eigenvalues, forward differences; and for zero eigenvalues we have at most an additional viscosity term of order $O(\Delta x)$. Hence the scheme is upwind and also the scheme is stable for $|\lambda| \Delta t / \Delta x \leqslant 1$.

Here we point out, that for (1) strictly hyperbolic, the eigenvalues of A are distinct real numbers, so $A-\lambda I$ is of rank $p-1$. For computation, we can use the cofactor subroutine to obtain R and R^{-1}, and then obtain sign$A$. Sometimes the elements of sign$A$ have simple analytic expressions, then they can be evaluated directly.

On the boundary, say the left boundary, let $F_j - F_{j-1} = 0$, the difference scheme becomes

$$\frac{\bar{U}_j^{n+1} - U_j^n}{\Delta t} + \frac{1}{2}(I - \operatorname{sign} A)\frac{F_{j+1}^n - F_j^n}{\Delta x} = 0, \tag{6}$$

where $\bar{U}_j^{n+1}$ denotes the preliminary boundary values obtained from this equation, A denotes $A_{j+\frac{1}{2}}^n$ or A_j^n. As above, with left multiplication by L, we arrive at

$$L\frac{\bar{U}_j^{n+1} - U_j^n}{\Delta t} + \frac{1}{2}(I - \operatorname{sign} A)\Lambda L\frac{U_{j+1}^n - U_j^n}{\Delta x} = 0,$$

i.e.

$$l\cdot\frac{\bar{U}_j^{n+1} - U_j^n}{\Delta t} = 0 \quad \text{for} \quad \lambda > 0, \tag{7a}$$

$$l\cdot\frac{\bar{U}_j^{n+1} - U_j^n}{\Delta t} + \lambda l\cdot\frac{U_{j+1}^n - U_j^n}{\Delta x} = 0 \quad \text{for} \quad \lambda < 0, \tag{7b}$$

$$l\cdot\frac{\bar{U}_j^{n+1} - U_j^n}{\Delta t} = 0 \quad \text{for} \quad \lambda = 0. \tag{7c}$$

For $\lambda < 0$, the characteristic is outgoing; $l\cdot\bar{U}_j^{n+1}$, with $\bar{U}_j^{n+1}$ obtained from (6), approximates $l\cdot U$ which would result from the corresponding characteristic relation. For $\lambda = 0$, the situation is similar. But for $\lambda > 0$, the characteristic is incoming, instead of the corresponding characteristic relation, a boundary condition should be given and used in computation. From $l\cdot\bar{U}_j^{n+1}$ and the boundary conditions, we can calculate U_j^{n+1} on the boundary.

Now, (7a) holds for steady-state problems. So for this type of problems, we can get preliminary values $\bar{U}_j^{n+1}$ from (6), and then modify these values with the given boundary conditions in an appropriate manner. The author also extended this boundary scheme to the two-dimensional case, see [6]. For the partial differential equation

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0,$$

the implicit scheme on the left boundary $x = 0$, say, is

$$\frac{U_{jk}^{n+1} - U_{jk}^n}{\Delta t} + \frac{1}{2}(I - \operatorname{sign} A)\left(\frac{F_{j+1k}^{n+1} - F_{jk}^{n+1}}{\Delta x} + \frac{G_{jk+1}^{n+1} - G_{jk-1}^{n+1}}{2\Delta y}\right) = 0,$$

here $A = \frac{\partial F}{\partial U}$. Then for $\lambda < 0$, the characteristic relation corresponds to that characteristic plane, whose normal projected on the xy plane is perpendicular to the boundary. For $\lambda = 0$, the corresponding characteristic plane is the plane tangent to the boundary and parallel to the t axis. The boundary scheme was applied to the calculation of steady-state axisymmetric transonic flow in nozzles in [6], the numerical method converged and the resulting steady-state solution was found to be quite reasonable. We note that the steady-state boundary difference scheme is in conservational form, which may be desirable for certain problems. Also, the implementation of this boundary scheme is very convenient, especially for implicit schemes; since for $\tilde{U}$, we can just set those parts of the coefficients and left-hand sides which have to do with the outside half of the equation to zero. Only after $\tilde{U}$ is obtained, need we concern ourselves with the special forms of the boundary conditions, see [6].

Many problems involve infinite regions, for numerical solution of these problems, it is necessary to reduce the infinite regions to finite regions. The boundary conditions given on the boundary of such a finite region is of utmost importance. For one-dimensional nonlinear hyperbolic systems with "no waves entering (the computational region) at $x=0$", Hedstrom in [7] gave the "non-reflecting" boundary condition

$$l\cdot\frac{\partial U}{\partial t}=0, \tag{8}$$

where l is the left eigenvector corresponding to $\lambda>0$. We see that (7a) is an approximation of just this "non-reflecting" boundary condition. Since boundary scheme (6) contains an approximation of (8), so with (6) we can obtain the approximations of all the $l\cdot U$, i. e. corresponding to all λ. Hence with (6), we can obtain directly U_j^{n+1} on the boundary.

In the following we apply upwind scheme (3) and boundary scheme (6) to the numerical solution of a Riemann problem in a finite computational region.

3. Numerical Example: Riemann Problem, "Non-Reflecting" Boundary Condition

The one-dimensional Euler equation is

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}=0,$$

where

$$U=\begin{pmatrix}\rho\\ \rho u\\ E\end{pmatrix},\quad F=\begin{pmatrix}\rho u\\ \rho u^2+p\\ (E+p)u\end{pmatrix},$$

$$p=(\gamma-1)\left(E-\frac{\rho u^2}{2}\right),$$

E is the energy per unit volume, the other letters have the usual meanings. It is well known that

$$A=\frac{\partial F}{\partial U}=\begin{pmatrix}0 & 1 & 0\\ \frac{\gamma-3}{2}u^2 & (3-\gamma)u & \gamma-1\\ (\gamma-1)u^3-\frac{\gamma Eu}{\rho} & \frac{3(1-\gamma)}{2}c^2+\frac{\gamma E}{\rho} & \gamma u\end{pmatrix},$$

and its eigenvalues are $u+c$, $u-c$, and u (c is the sonic velocity). For $u+c>0$, $u-c<0$, $u>0$, it can be shown that

$$\operatorname{sign}A=\begin{pmatrix} 1-\frac{u}{c}-\frac{\gamma-1}{2}\frac{u^2}{c^2} & \frac{1}{c}+(\gamma-1)\frac{u}{c^2} & -\frac{\gamma-1}{c^2} \\ u+\frac{\gamma-3}{2}\frac{u^2}{c}-\frac{\gamma-1}{2}\frac{u^3}{c^2} & -(\gamma-2)\frac{u}{c}+(\gamma-1)\frac{u^2}{c^2} & \frac{\gamma-1}{c}-(\gamma-1)\frac{u}{c^2} \\ \frac{u^2}{2}+\frac{\gamma-2}{2}\frac{u^3}{c}-\frac{\gamma-1}{4}\frac{u^4}{c^2}-\frac{uc}{\gamma-1} & \frac{-(2\gamma-3)}{2}\frac{u^2}{c}+\frac{\gamma-1}{2}\frac{u^3}{c^2}+\frac{c}{\gamma-1} & (\gamma-1)\frac{u}{c}-\frac{\gamma-1}{2}\frac{u^2}{c^2} \end{pmatrix}.$$

For supersonic flow where also $u-c>0$, $\operatorname{sign}A$ is simply I.

Let us consider the shock tube problem as shown in Fig. 1, i. e. a Riemann problem, with initial values taken from Sod [8],

$$u_L=0, \qquad u_R=0,$$
$$p_L=1, \qquad p_R=0.1,$$
$$\rho_L=1, \qquad \rho_R=0.125.$$

Consider also the finite computational region as shown in Fig. 1; on the boundaries we give the "non-reflecting" boundary condition (8). Now using the upwind difference scheme (3), the left boundary scheme (6), and the corresponding right boundary scheme, we get directly the numerical solution of this problem. For the above initial values, the numerical results with $\Delta x=0.01$, $q=\frac{\Delta t}{\Delta x}=0.44231$ (which satisfies the CFL condition and at the same time allows the printed results to be compared with those of Sod) are shown in Fig. 2—5 by "·". We see that: (1) With the upwind scheme (3), there are about three points of transition for the unsteady shock (Fig. 2a, 5a), which is not bad for a first order scheme. (2) When the rarefaction wave and the shock wave go out of the computational region, there are no apparent reflections from the boundaries and no obvious deviations of any sort near the boundaries (Fig. 2b—5b). This result is better than that obtained by Hedstrom in his numerical example.

But as with other schemes, the width of the contact discontinuity spreads with increasing n. Hence for $x>0$, "artificial compression" (AC) as proposed by Harten [9] was added at each time level. The numerical results are indicated in the figures by "×", where there is apparent difference from that without AC. The value for q in AC was also taken to be 0.44231, though it can go up to 1. We see that: (3) By adding artificial compression at each time level, there is about one point of transition for the shock; the width of the contact discontinuity does not increase with n; and there are no apparent reflections from the boundaries, the numerical

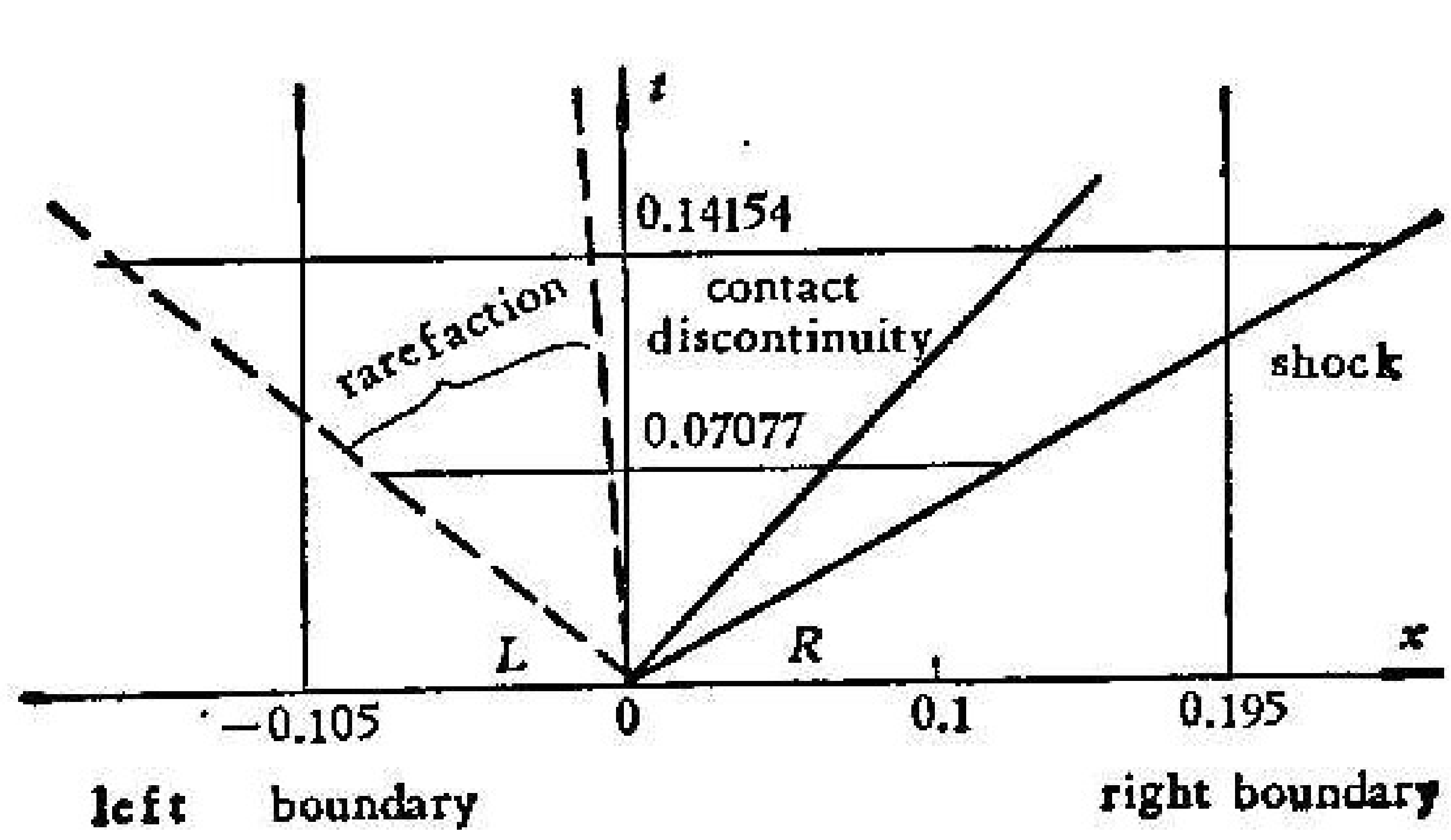

Fig. 1. Shock tube problem

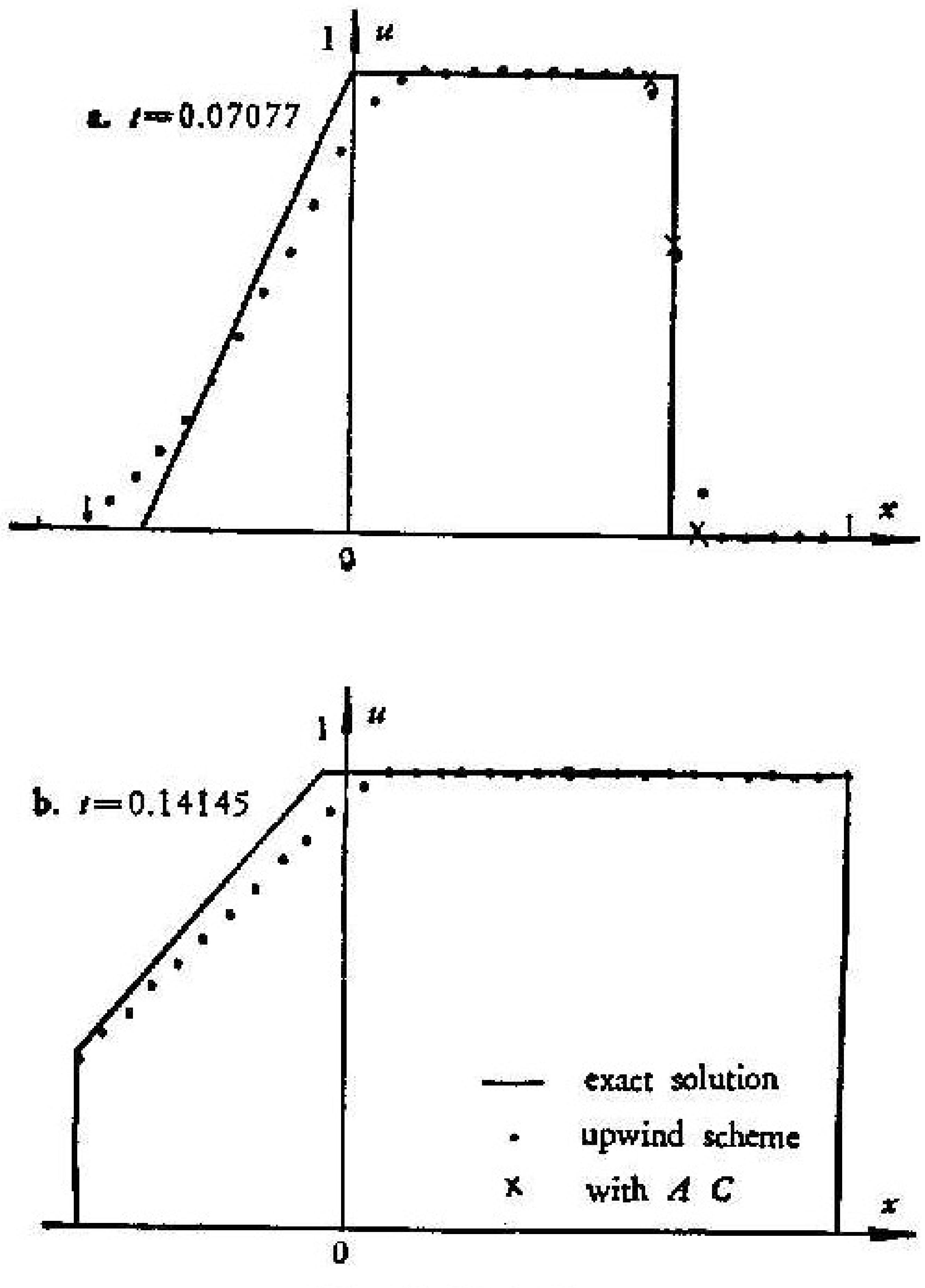

Fig. 2. Velocity u

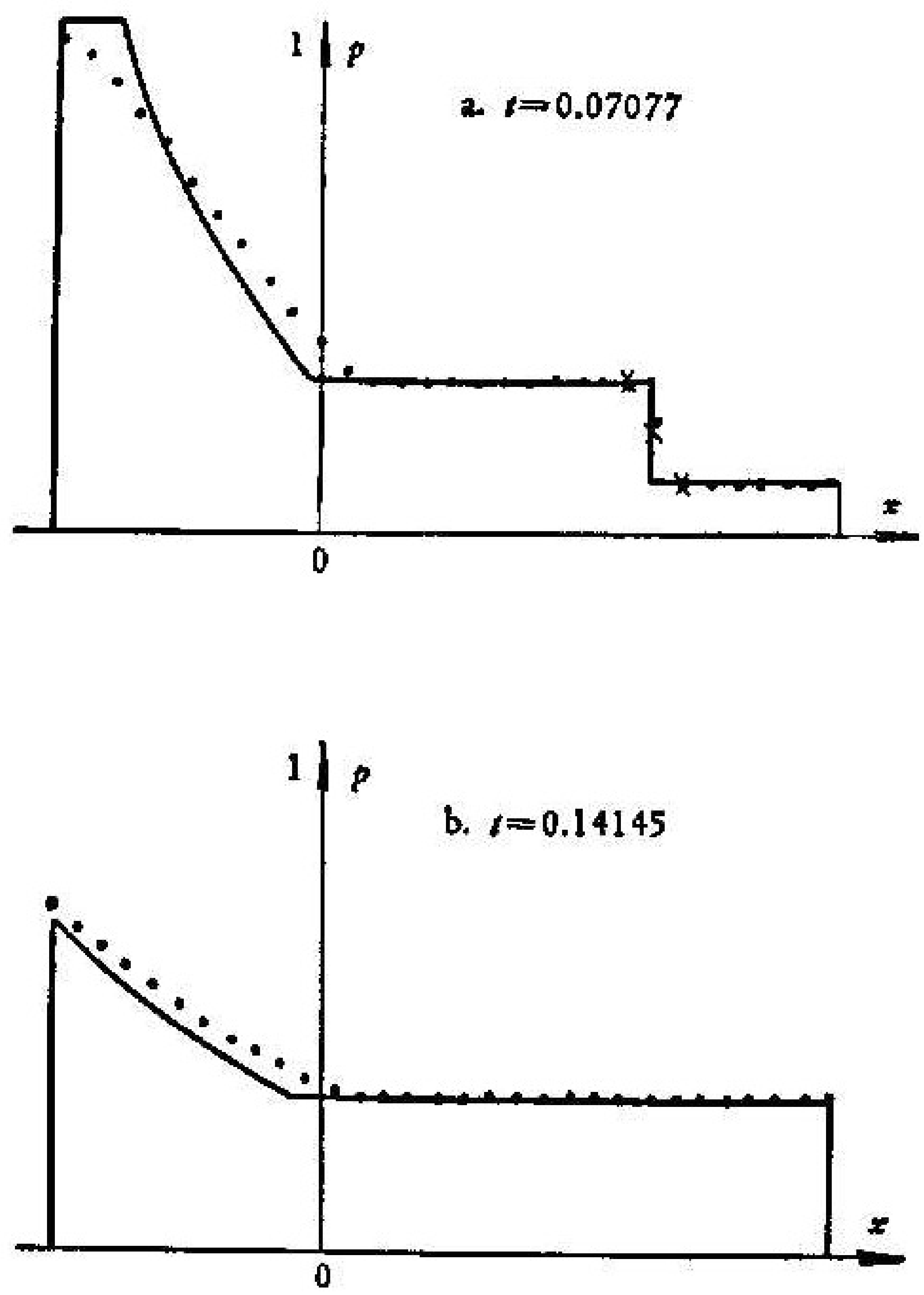

Fig. 3. Pressure p

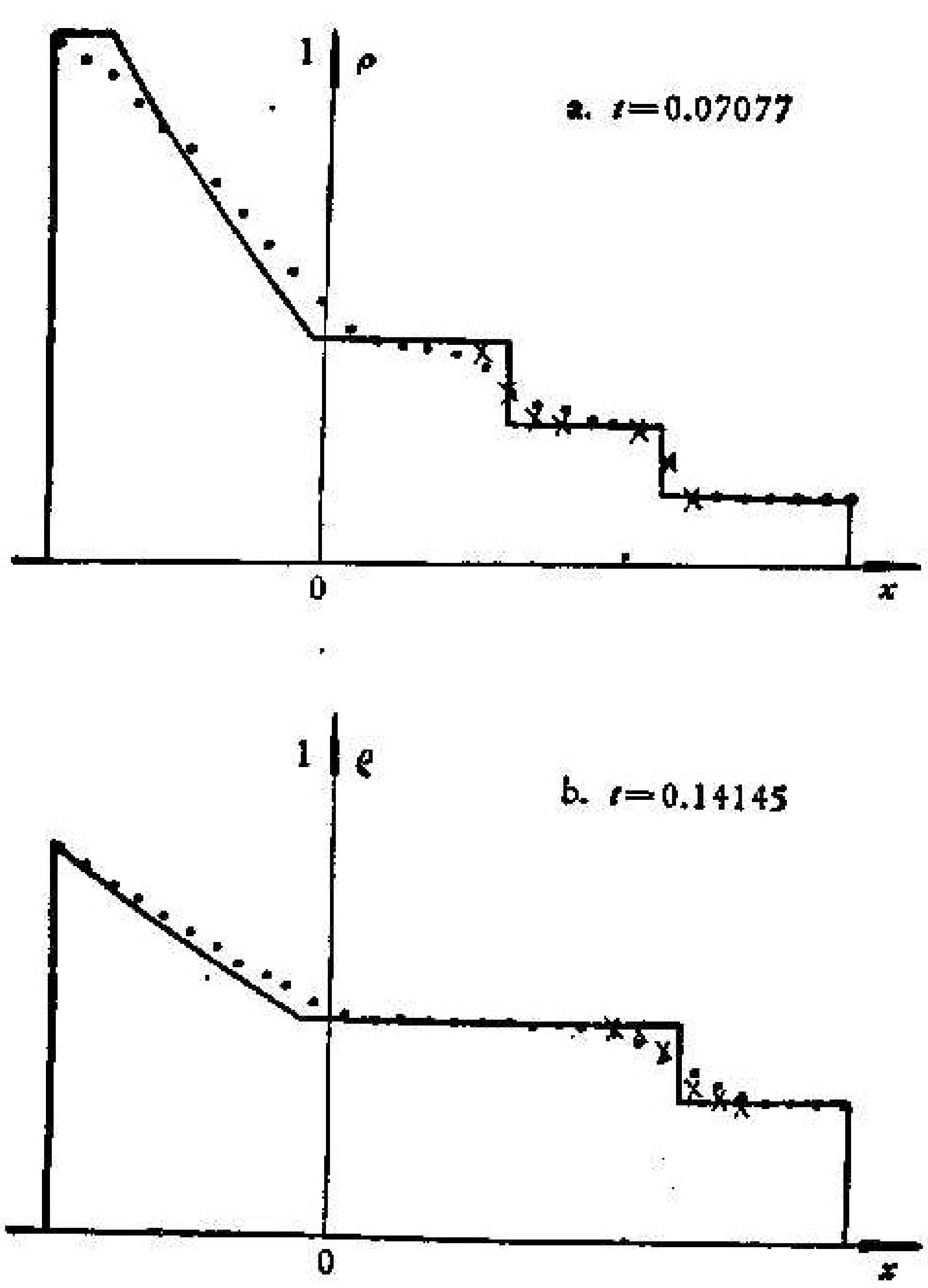

Fig. 4. Density ρ

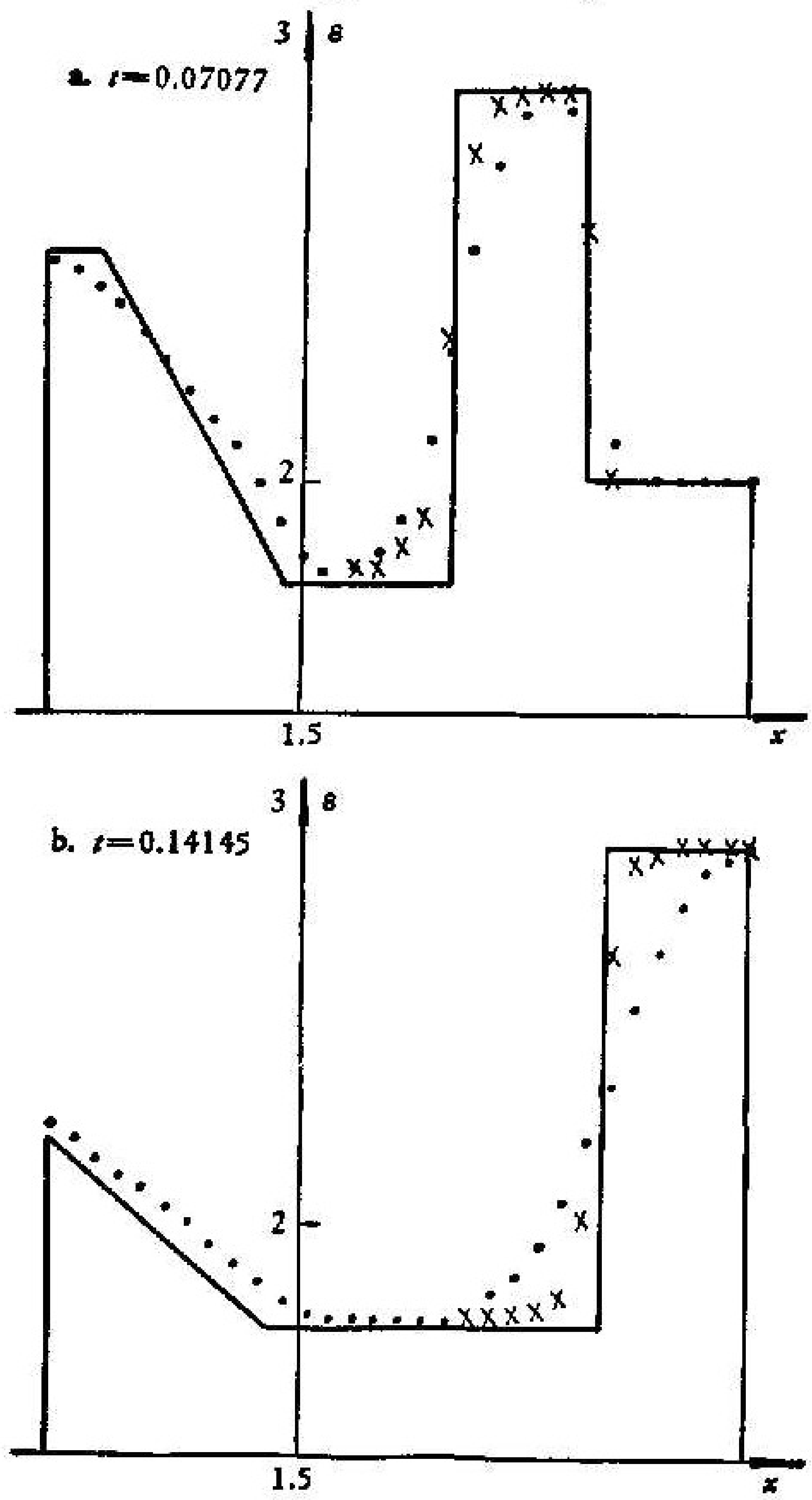

Fig. 5. Internal energy e

solution remains very cleancut near the boundaries.

The results of this numerical example are promising. Further attempts to apply this upwind scheme and the corresponding boundary scheme to other problems will be made.

References

[1] R. Courant, E. Isaacson, and M. Rees, On the solution of nonlinear hyperbolic differential equations by finite differences, *Comm. Pure Appl. Math.*, 5 (1952), 243—256.

[2] R. D. Richtmyer, K. W. Morton, Difference methods for initial value problems, Interscience Publishers, Inc., New York, 1967.

[3] B. Engquist, S. Osher, Stable and entropy satisfying approximation for transonic flow calculation, *Math, Comput.*, 34 (1980), 45—75.

[4] J. L.. Steger, R. F. Warming, Flux vector splitting of the inviscid gasdynamic equations with application to finite-difference methods, *J. Comput. Phys.*, 40 (1981), 263—294.

[5] L. C. Huang, Pseudo-unsteady difference schemes for discontinuous solutions of steady-state, one-dimensional fluid dynamic problems, *J. Comput. Phys.*, 42 (1981), 195—211.

[6] L. C. Huang, A boundary difference scheme and its application in the computation of steady-state axisymmetric flow in nozzles, *Acta Aerodynamica Sinica*, 3 (1982), 54—61. (In Chinese)

[7] G. W. Hedstrom, Non-reflecting boundary condition for nonlinear hyperbolic systems, *J. Comput. Phys.*, 30 (1979), 222—237.

[8] G. A. Sod, A survey of several finite difference methods for the system of nonlinear hyperbolic conservational laws, *J. Comput. Phys.*, 27 (1978), 1—31.

[9] A. Harten, The artificial compression method for computation of shocks and contact discontinuities III. Self-adjusting hybrid, *Math. Comput.*, 32 (1978), 363—390.

[10] G. Q. Zhang, A difference scheme for the aerodynamic equations, *Applied Mathematics and Computational Mathematics*, 1 (1964), 57—68. (In Chinese)