A FAMILY OF STIFFLY STABLE LINEAR MULTISTEP METHODS FOR STIFF AND HIGHLY OSCILLATORY ORDINARY DIFFERENTIAL EQUATIONS*

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Abstract

This paper suggests a family of stiffly stable linear k-step methods with order k, for arbitrary k. Their stability regions are larger than those of the Gear method[1]. Preliminary numerical test shows that these methods are efficient for stiff systems of ordinary differential equations with characteristic values near the imaginary axis.

1. Introduction

In [2] the author has constructed three families of linear k-step methods, depending on parameter $\varepsilon>0$, with good stability. The three families of methods are:

1) asymptotically A-stable[1)] implicit linear k-step methods with order $k+1$, which have the generating polynomials[3]

$$\rho_\varepsilon(\xi)=(\xi-1)(\xi-1+\varepsilon)^{k-1},$$
$$\sigma_\varepsilon(\xi)=c_0+c_1(\xi-1)+\cdots+c_{k-1}(\xi-1)^{k-1}+c_k(\xi-1)^k. \tag{1}$$

where c_i are determined by the relationship $\frac{\rho_\varepsilon(\xi)}{\ln\xi}=c_0+c_1(\xi-1)+\cdots+c_k(\xi-1)^k+\cdots$, and so are the following $c_i(i=1,\cdots,k)$.

2) stiffly stable, asymptotically A-stable implicit linear k-step methods with order k, which have the generating polynomials

$$\rho_\varepsilon(\xi)=(\xi-1)(\xi-1+\varepsilon)^{k-1},$$
$$\sigma_\varepsilon(\xi)=c_0+c_1(\xi-1)+\cdots+c_{k-1}(\xi-1)^{k-1}+p(\xi-1)^k,\ \frac{1}{2}<p<\infty. \tag{2}$$

This family of methods involves two parameters ε and p; when p is chosen in $\left(\frac{1}{2},\ \infty\right)$, the subfamily of methods is stiffly stable and asymptotically stable as $\varepsilon\to0$.

3)[2)] asymptotically A-stable explicit linear k-step methods with order $k-1$, which have the generating polynomials

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1) A family of methods $\{M(\varepsilon)\}$ depending on parameter $\varepsilon>0$ is called asymptotically A-stable if for any $R>0$, $0\leqslant\alpha<\frac{\pi}{2}$, we can find $\varepsilon_0>0$, such that when $\varepsilon<\varepsilon_0$, $M(\varepsilon)$ is stable in the region $\Omega_{\alpha,R}$, where

$$\Omega_{\alpha,R}=\{\mu\in\mathbb{C}\mid|\mu|\leqslant R,\ |\arg(-\mu)|\leqslant\alpha\}.$$

2) An asymptotically A-stable family of explicit linear k-step methods with order $k-1$ has already been constructed in [4]; however, the family of methods mentioned here is different from that one.

$$\rho_s(\xi) = (\xi-1)(\xi-1+s)^{k-1},$$
$$\sigma_s(\xi) = c_0 + c_1(\xi-1) + \cdots + c_{k-2}(\xi-1)^{k-2} + p(\xi-1)^{k-1},\ p \in R \tag{3}$$

when $p \to 0$ and $\frac{s}{p} \to 0$, this family of methods is asymptotically A-stable.

It is shown in [2] that the implicit linear k-step methods of order k with the generating polynomials

$$\rho_s(\xi) = (\xi-1)(\xi-1+s)^{k-1},$$
$$\sigma_s(\xi) = c_0 + c_1(\xi-1) + \cdots + c_{k-1}(\xi-1)^{k-1} + c_k^*(\xi-1)^k, \tag{4}$$

where $\quad c_k^* = c_{k-1} - c_{k-2} + \cdots + (-1)^{k-1} c_0$

are also stiffly stable and asymptotically A-stable.

Because all these families of methods are asymptotically A-stable, we can expect to find linear multistep methods with good stability properties from any of them.

In this paper a family of stiffly stable linear multistep methods with orders one to six is obtained from (4), whose stability regions are larger than those of Gear method.

2. A Family of Stiffly Stable Linear Multistep Methods with Orders One to Six

If we choose (4) as generating polynomials, it is very simple to write down the implicit linear k-step methods with orders one to six. From now on, we denote these linear multistep methods with order k ($k=1, 2, \cdots, 6$) depending on s by $M_k(s)$ and Gear method with order k by G_k for short.

Now we list these formulas and major parameters of their stability regions in the following Tables 1—4. (In the same tables we also list corresponding paramaters of Gear formulas for comparison, and the meaning of the parameters D and α characterizing the stiff stability are shown by Fig. 1.) These tables show that their stability regions are much larger than those of Gear method.

It is convenient to describe a linear k-step method $\sum_0^k \alpha_i Y_{n+i} = h \sum_0^k \beta_i f_{n+i}$ by its generating polynomials $\rho(\xi) = \alpha_k \xi^k + \alpha_{k-1}\xi^{k-1} + \cdots + \alpha_0$ and $\sigma(\xi) = \beta_k \xi^k + \beta_{k-1}\xi^{k-1} + \cdots + \beta_0$. Therefore we only write down $\rho(\xi)$ and $\sigma(\xi)$ for corresponding linear multistep

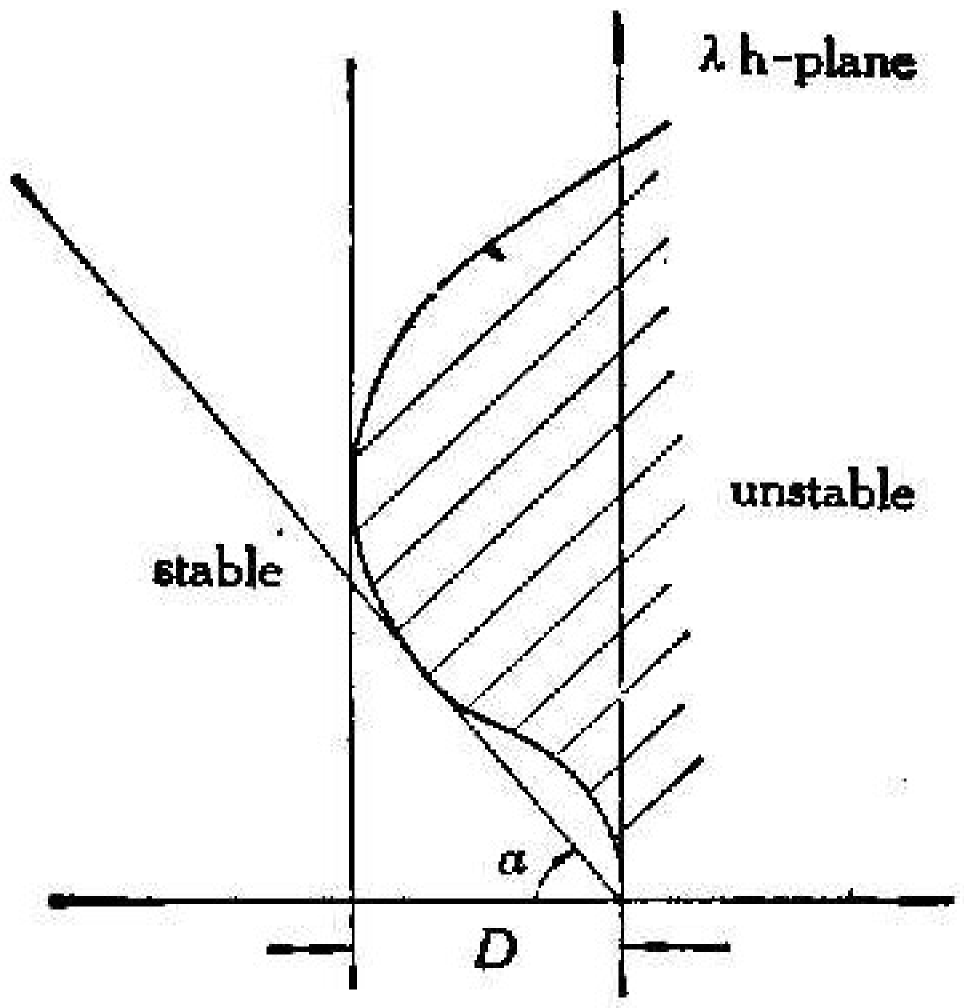

Fig. 1

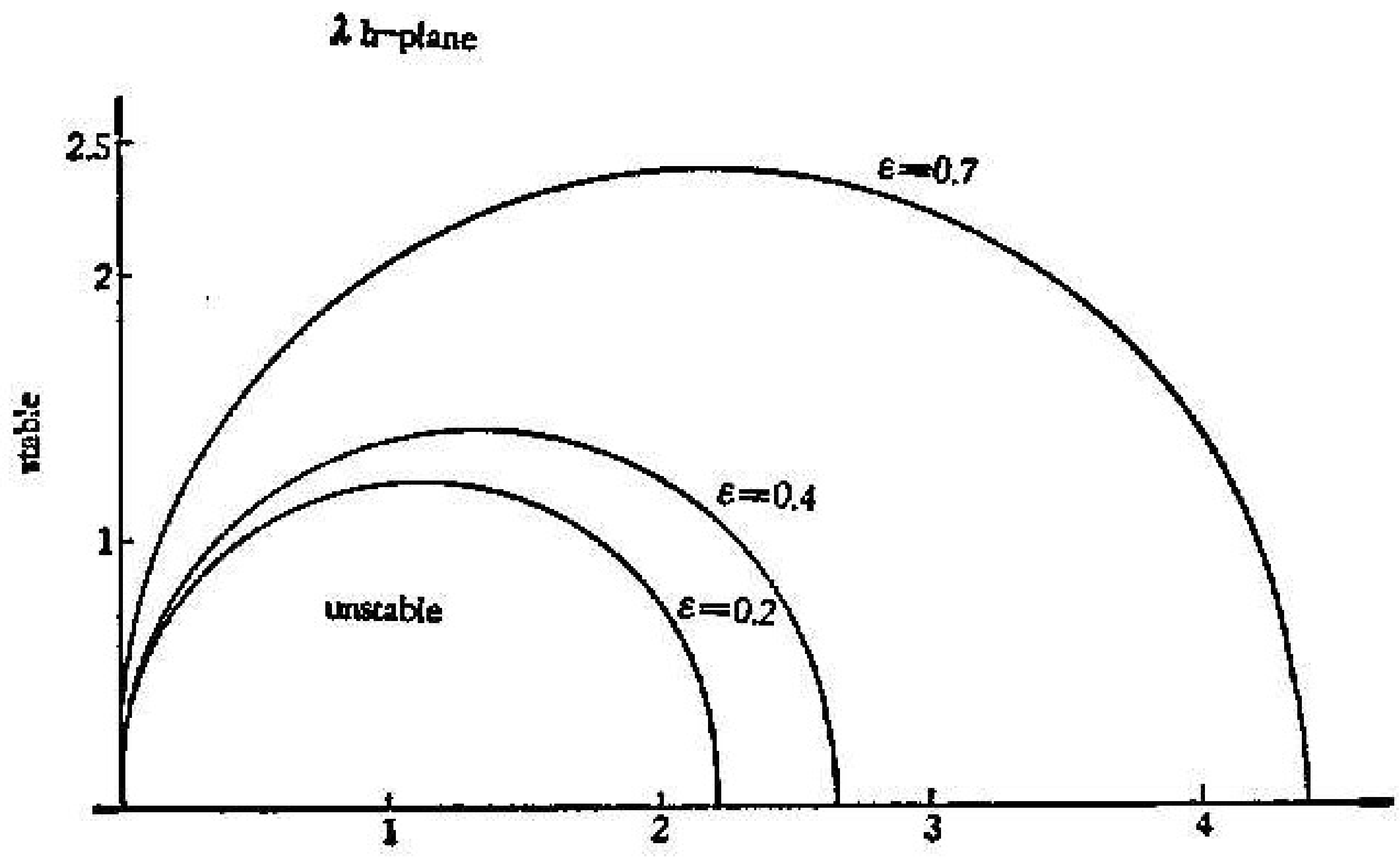

Fig. 2

method in the following.

Methods $M_2(\varepsilon)$:

$$\rho_\varepsilon(\xi)=\xi^2-(2-\varepsilon)\xi+(1-\varepsilon),$$

$$\sigma_\varepsilon(\xi)=\left(1-\frac{\varepsilon}{2}\right)\xi^2-\left(1-\frac{3\varepsilon}{2}\right)\xi.$$

Similar to the G_2, the methods $M_2(\varepsilon)$ are also A-stable when $\varepsilon<0.7$ by numerical computation, see Fig. 2.

Methods $M_3(\varepsilon)$:

$$\rho_\varepsilon(\xi)=\xi^3-(3-2\varepsilon)\xi^2+(3-4\varepsilon+\varepsilon^2)\xi-(1-\varepsilon)^2,$$

$$\sigma_\varepsilon(\xi)=\left(1-\varepsilon+\frac{5}{12}\varepsilon^2\right)\xi^3-\left(2-4\varepsilon+\frac{4}{3}\varepsilon^2\right)\xi^2+\left(1-3\varepsilon+\frac{23}{12}\varepsilon^2\right)\xi.$$

Table 1

	G_3	$M_3(0.7)$	$M_3(0.6)$	$M_3(0.5)$	$M_3(0.4)$	$M_3(0.3)$	$M_3(0.2)$	$M_3(0.1)$	$M_3(0.04)$	$M_3(0.01)$
D	0.1	0.0735	0.0298	0.015	0.0076	0.0036	0.0013	0.24^{-3}	0.47^{-4}	~ 0
α	$\sim 80°$	$\sim 87.3°$	$\sim 88.23°$	$\sim 88.68°$	$\sim 89.07°$	$\sim 89.29°$	$\sim 89.6°$	$\sim 89.85°$	$\sim 89.91°$	$\sim 90°$

Methods $M_4(\varepsilon)$:

$$\rho_\varepsilon(\xi)=\xi^4-(4-3\varepsilon)\xi^3+3(2-3\varepsilon+\varepsilon^2)\xi^2-(4-9\varepsilon+6\varepsilon^2-\varepsilon^3)\xi+(1-\varepsilon)^3,$$

$$\sigma_\varepsilon(\xi)=\left(1-\frac{3}{2}\varepsilon+\frac{5}{4}\varepsilon^2-\frac{3}{8}\varepsilon^3\right)\xi^4-\left(3-\frac{15}{2}\varepsilon+\frac{21}{4}\varepsilon^2-\frac{37}{24}\varepsilon^3\right)\xi^3$$
$$+\left(3-\frac{21}{2}\varepsilon+\frac{39}{4}\varepsilon^2-\frac{59}{24}\varepsilon^3\right)\xi^2-\left(1-\frac{9}{2}\varepsilon+\frac{23}{4}\varepsilon^2-\frac{55}{24}\varepsilon^3\right)\xi.$$

Table 2

	G_4	$M_4(0.5)$	$M_4(0.4)$	$M_4(0.3)$	$M_4(0.2)$	$M_4(0.1)$	$M_4(0.04)$	$M_4(0.01)$
D	0.7	0.103	0.0465	0.0213	0.00807	0.0017	0.19^{-3}	~ 0
α	$\sim 73°$	$\sim 85.85°$	$\sim 86.9°$	$\sim 87.78°$	$\sim 88.6°$	$\sim 89.23°$	$\sim 89.82°$	$\sim 90°$

Methods $M_5(\varepsilon)$:

$$\rho_\varepsilon(\xi)=\xi^5-(5-4\varepsilon)\xi^4+2(5-8\varepsilon+3\varepsilon^2)\xi^3-2(5-12\varepsilon+9\varepsilon^2-2\varepsilon^3)\xi^2$$
$$+(5-16\varepsilon+18\varepsilon^2-8\varepsilon^3+\varepsilon^4)\xi-(1-\varepsilon)^4,$$

$$\sigma_\varepsilon(\xi)=\left(1-2\varepsilon+\frac{5}{2}\varepsilon^2-\frac{3}{2}\varepsilon^3+\frac{251}{720}\varepsilon^4\right)\xi^5-\left(4-12\varepsilon+13\varepsilon^2-\frac{23}{3}\varepsilon^3+\frac{637}{360}\varepsilon^4\right)\xi^4$$
$$+\left(6-24\varepsilon+30\varepsilon^2-16\varepsilon^3+\frac{109}{30}\varepsilon^4\right)\xi^3-\left(4-20\varepsilon+31\varepsilon^2-19\varepsilon^3+\frac{1387}{360}\varepsilon^4\right)\xi^2$$
$$+\left(1-6\varepsilon+\frac{23}{2}\varepsilon^2-\frac{55}{6}\varepsilon^3+\frac{1901}{720}\varepsilon^4\right)\xi.$$

Table 3

	G_5	$M_5(0.4)$	$M_5(0.3)$	$M_5(0.2)$	$M_5(0.1)$	$M_5(0.04)$	$M_5(0.01)$
D	2.4	0.283	0.091	0.009	0.216^{-2}	0.33^{-3}	~ 0
α	$\sim 63°$	$\sim 79.58°$	$\sim 84.12°$	$\sim 86°$	$\sim 88.08°$	$\sim 89.4°$	$\sim 90°$

Methods $M_6(s)$:

$$\rho_s(\xi)=\xi^6-(6-5s)\xi^5+5(3-5s+2s^2)\xi^4-10(2-5s+4s^2-s^3)\xi^3$$
$$+5(3-10s+12s^2-6s^3+s^4)\xi^2$$
$$-(6-25s+40s^2-30s^3+10s^4-s^5)\xi+(1-s)^5,$$

$$\sigma_s(\xi)=\left(1-\frac{5}{2}s+\frac{25}{6}s^2-\frac{45}{12}s^3+\frac{251}{144}s^4-\frac{1425}{4320}s^5\right)\xi^6$$
$$-\left(5-\frac{35}{2}s+\frac{155}{6}s^2-\frac{275}{12}s^3+\frac{1525}{144}s^4-\frac{8631}{4320}s^5\right)\xi^5$$
$$+\left(10-45s+\frac{215}{3}s^2-\frac{355}{6}s^3+\frac{1945}{72}s^4-\frac{3649}{720}s^5\right)\xi^4$$
$$-\left(10-55s+\frac{305}{3}s^2-\frac{175}{2}s^3+\frac{2695}{72}s^4-\frac{4991}{720}s^5\right)\xi^3$$
$$+\left(5-\frac{65}{2}s+\frac{425}{6}s^2-\frac{845}{12}s^3+\frac{4675}{144}s^4-\frac{2641}{480}s^5\right)\xi^2$$
$$-\left(1-\frac{15}{2}s+\frac{115}{6}s^2-\frac{275}{12}s^3+\frac{1901}{144}s^4-\frac{4277}{1440}s^5\right)\xi.$$

Table 4

	G_6	$M_6(0.4)$	$M_6(0.3)$	$M_6(0.2)$	$M_6(0.1)$	$M_6(0.04)$	$M_6(0.01)$	$M_6(0.001)$
D	6.1	0.363	0.116	0.041	0.89^{-2}	0.133^{-2}	0.52^{-3}	0.4^{-3}
α	~27°	~82.16°	~84.29°	~86°	~88.09°	~89.21°	~89.04°	~89.28°

3. Discussion

The stability region of a linear k-step method $(\rho(\xi),\ \sigma(\xi))$ only tells us that, if λh falls into it, then all the roots $\xi_j(\lambda h)$ $(j=1,\ \cdots,\ k)$ of the characteristic equation

$$\rho(\xi)=\lambda h\sigma(\xi) \tag{5}$$

fall into the unit circle, and the errors of the method is decreasing. Otherwise the method is divergent. For chosen stepsize h and any characteristic root λ_i of the linear system, we generally do not know the distribution of the roots $\xi_j(\lambda_i h)$ $(j=1,\ \cdots,\ k)$ of the characteristic equation corresponding to such a $\lambda_i h$ in the unit circle. Obviously the closer $\xi_j(\lambda_i h)$ come to zero the faster the errors decrease. Conversely the closer the roots come to 1 the slower the errors decrease. Therefore it is insufficient to compare strictly two methods merely by their stable regions. However, the information about the distribution of roots of the generating polynomials $\rho(\xi)=0$, $\sigma(\xi)=0$ seems significant for comparison. It may be used as a crude measure.

We give the reason as follows.

As $\lambda h\to 0$, (5) becomes $\rho(\xi)=0$, and as $\lambda h\to\infty$, (5) becomes $\sigma(\xi)=0$. Owing to continuity, all the roots of (5) are near to the roots of the equation $\rho(\xi)=0$ when $\lambda h\sim 0$, and near to the roots of $\sigma(\xi)=0$ when $\lambda h\sim\infty$.

For obtaining certain precision in calculation. we always choose at the start $\lambda h\ll 1$. Therefore the method has better behavior if, except for the major root $\xi=1$, the other roots of $\rho(\xi)=0$ in modulus are smaller. Adams formulas are advantageous in this respect.

In solving stiff equations, we always increase the stepsize successively as soon as the transient process caused by the largest (in modulus) characteristic values passes over. Then it may happen that $|\lambda|_{max}h \gg 1$, and the roots of (5) tend to the roots of $\sigma(\xi)=0$. Similarly, the method has better behavior if the roots of $\sigma(\xi)=0$ in modulus are smaller. Gear formulas are advantageous in this respect.

Now we illustrate these by an example. We intergrate the differential equation (6) using the linear multistep formula.

$$\begin{cases} y_1' = \lambda_1 y_1, \\ y_2' = \lambda_2 y_2, \\ y_3' = \lambda_3 y_3, \end{cases} \tag{6}$$

where $\mathrm{Re}\lambda_i<0$ and $|\lambda_1| \gg |\lambda_2| > |\lambda_3|$.

First, we should begin with stepsize h_1 satisfying $h_1|\lambda_1| \ll 1$ (thus, $h_1|\lambda_2| \ll 1$, $h_1|\lambda_3| \ll 1$). Then the error propagation is influenced mainly by the distribution of the roots of $\rho(\xi)=0$.

When $e^{\lambda_1 t} \sim 0$, we should increase stepsize to h_2, such that $h_2|\lambda_2| \ll 1$ (thus, $h_2|\lambda_3| \ll 1$, but it may happen that $h_2|\lambda_1| \gg 1$). Then both the distributions of the roots of $\rho(\xi)=0$, $\sigma(\xi)=0$ influence the error propagation.

Moreover when $e^{\lambda_2 t} \sim 0$, we should again choose stepsize h_3, such that $h_3|\lambda_3| \ll 1$ (thus $h_3|\lambda_1| \gg 1$). Then all the distributions of the roots of $\rho(\xi)=0$, $\sigma(\xi)=0$ and $\rho(\xi)=\lambda_3 h_3 \sigma(\xi)$ influence the error propagation.

Therefore, the distributions of the roots of the $\rho(\xi)=0$ and $\sigma(\xi)=0$ should be considered if we want to compare Gear method with methods $M(\varepsilon)$.

The distribution of roots of $\rho(\xi)=0$ and $\sigma(\xi)=0$ corresponding to G_2, G_3 and methods $M_2(\varepsilon)$, $M_3(\varepsilon)$ are listed in the following tables.

Table 5

	G_2	$M_2(0.7)$	$M_2(0.8)$
roots of $\rho(\xi)$	$1, \frac{1}{3}$	1, 0.3	1, 0.2
roots of $\sigma(\xi)$	0, 0	0, −0.077	0, −0.333

Table 6

	G_3	$M_3(0.6)$	$M_3(0.7)$
roots of $\rho(\xi)$	$1, 0.318 \pm 0.284i$	1, 0.4, 0.4	1, 0.3, 0.3
roots of $\sigma(\xi)$	0, 0, 0	−0.38, 0.526, 0	−0.728, 0.438, 0

It seems that Gear method is not notably superior to the methods $M(\varepsilon)$ with respect to the distributions of roots of $\rho(\xi)=0$ and $\sigma(\xi)=0$ for the examples mentioned above.

4. Numerical Tests

The well-known model stiff system of linear ordinary differential equations (A) is solved using both Gear method of order 4 and the methods $M(\varepsilon)$ with the same order ($\varepsilon=0.6$, 0.5, 0.4, 0.3, 0.2). We choose $h=0.01$, the start point $t_0=1$ (so the early transients have past over) and the end point $t_f=10$. The parameter α in (A) is taken to be 25, 100, 200, 300 and 700 respectively. The computational results in Tables 7–11 show that:

1) When $\alpha=25$, Gear method can proceed smoothly, and so does the method $M(\varepsilon)$. The precision of both methods is almost the same. (see Table 7)

Table 7

$\alpha=25$, $t=t_f$	exact solution	G_4	$M_4(0.6)$	$M_4(0.5)$
y_1	0	-0.8738041831^{-38}	0.4613560662^{-38}	0.5676106023^{-38}
y_2	0	-0.9884929319^{-38}	0.3746888796^{-38}	0
y_3	0.4248354378^{-17}	0.4248270560^{-17}	0.4248190658^{-17}	0.4247936542^{-17}
y_4	0.4539993009^{-4}	0.4539992863^{-4}	0.4539992783^{-4}	0.4539992320^{-4}
y_5	0.6737947023^{-2}	0.6737946963^{-2}	0.6737946990^{-2}	0.6737946675^{-2}
y_6	0.3678794414^{0}	0.3678794385^{0}	0.3678784689^{0}	0.3678794215^{0}

$\alpha=25$, $t=t_f$	$M_4(0.4)$	$M_4(0.3)$	$M_4(0.2)$
y_1	-0.3244391067^{-37}	-0.8008500600^{-38}	0.7171939749^{-35}
y_2	-0.2346312179^{-37}	-0.2198759832^{-38}	0.7429037992^{-35}
y_3	0.4247136262^{-17}	0.4246931980^{-17}	0.4223131740^{-17}
y_4	0.4539991463^{-4}	0.4539989904^{-4}	0.4539978429^{-4}
y_5	0.6737946203^{-2}	0.6737947841^{-2}	0.6737948122^{-2}
y_6	0.3678793896^{0}	0.3678794856^{0}	0.3678795296^{0}

2) When $\alpha=100$, Gear method becomes unstable, but the methods $M(\varepsilon)$ still succeed with high precision. (see Table 8)

Table 8

$\alpha=100$, $t=t_f$	exact solution	G_4	$M_4(0.6)$	$M_4(0.5)$
y_1	0	0.4144459529^{9}	-0.4258798578^{-8}	0.6521020263^{-9}
y_2	0	0.9916949711^{10}	0.1556228363^{-8}	0.4642476414^{-9}
y_3	0.4248354378^{-17}	0.4248270560^{-17}	0.4248190658^{-17}	0.4247936642^{-17}
y_4	0.4539993009^{-4}	0.4539992863^{-4}	0.4539992783^{-4}	0.4539992320^{-4}
y_5	0.6737947023^{-2}	0.6737946963^{-2}	0.6737946990^{-2}	0.6737946675^{-2}
y_6	0.3678794414^{0}	0.3678794385^{0}	0.3678794389^{0}	0.3678794215^{0}

$\alpha=100$, $t=t_f$	$M_4(0.4)$	$M_4(0.3)$	$M_4(0.2)$
y_1	0.1314629451^{-16}	-0.1201952788^{-33}	-0.2146341723^{-38}
y_2	0.7330294851^{-17}	-0.6112930319^{-34}	-0.3087916265^{-38}
y_3	0.4247136262^{-17}	0.4243931980^{-17}	0.4223131740^{-17}
y_4	0.4539991463^{-4}	0.4539989904^{-4}	0.4539978429^{-4}
y_5	0.6737946203^{-2}	0.6737947841^{-2}	0.6737948122^{-2}
y_6	0.3678793898^{0}	0.3678794856^{0}	0.3678795296^{0}

3) When $\alpha=200$ and 300, both Gear method and the method $M(0.6)$ loss stability, but the methods $M(s)$ ($s=0.5$, 0.4, 0.3, 0.2) still succeed with high precision. (see Table 9 and 10)

Table 9

$\alpha=200$ $t=t_f$	exact solution	G_4	$M_4(0.6)$	$M_4(0.5)$
y_1	0		-0.3949833340^{23}	0.2767585889^{-11}
y_2	0	y_1, y_2	0.4262849250^{22}	0.5785548813^{-11}
y_3	0.4248354378^{-17}		0.4248190658^{-17}	0.4247926542^{-17}
y_4	0.4539993009^{-4}	overflow	0.4539992783^{-4}	0.4539992320^{-4}
y_5	0.6737947023^{-2}		0.6737946990^{-2}	0.6737946675^{-2}
y_6	0.3678794414^{0}		0.3678794689^{0}	0.3678794215^{0}

$\alpha=200$ $t=t_f$	$M_4(0.4)$	$M_4(0.3)$	$M_4(0.2)$
y_1	0.7393149667^{-38}	0.5812502772^{-38}	-0.1143503665^{-37}
y_2	0.2472637221^{-38}	-0.7119410453^{-38}	0.1589647785^{-37}
y_3	0.4247136262^{-17}	0.4243931980^{-17}	0.4223131740^{-17}
y_4	0.4539991463^{-4}	0.4539989904^{-4}	0.4539978429^{-4}
y_5	0.6737946203^{-2}	0.6737947841^{-2}	0.6737948122^{-2}
y_6	0.3678793896^{0}	0.3678794856^{0}	0.3678795296^{0}

Table 10

$\alpha=300$ $t=t_f$	exact solution	G_4	$M_4(0.6)$	$M_4(0.5)$
y_1	0		-0.7091651982^{22}	0
y_2	0	y_1, y_2	0.2900400791^{22}	0
y_3	0.4248354378^{-17}		0.4248190658^{-17}	0.4247936542^{-17}
y_4	0.4539993009^{-4}		0.4539992783^{-4}	0.4539992320^{-4}
y_5	0.6737947023^{-2}	overflow	0.6737946990^{-2}	0.6737946675^{-2}
y_6	0.3678794414^{0}		0.3678794689^{0}	0.3678794215^{0}

$\alpha=300$ $t=t_f$	$M_4(0.4)$	$M_4(0.3)$	$M_4(0.2)$
y_1	0	0.9962676100^{-38}	-0.1817668606^{-37}
y_2	0	0.5422554094^{-38}	-0.8716860559^{-38}
y_3	0.4247136262^{-17}	0.4243931980^{-17}	0.4223131740^{-17}
y_4	0.4539991463^{-4}	0.4539989904^{-4}	0.4539978429^{-4}
y_5	0.6737946203^{-2}	0.6737947841^{-2}	0.6737948122^{-2}
y_6	0.3678793896^{0}	0.3678794856^{0}	0.3678795296^{0}

4) When $\alpha=700$, using the method $M(0.2)$ we also successfully obtain high precision solution. (see Table 11)

Table 11

$\alpha=700,\ t=t_f$	y_1	y_2	y_3
exact solution	0	0	0.4248354378^{-17}
$M_4(0.2)$	-0.1118689592^{-37}	-0.2299207697^{-38}	0.4223131740^{-17}
$\alpha=700,\ t=t_f$	y_4	y_5	y_6
exact solution	0.4539993009^{-2}	0.6737947023^{-2}	0.3678794414^{0}
$M_4(0.2)$	0.4539978429^{-4}	0.6737948122^{-2}	0.3678795296^{0}

5) Among the methods $M(s)$, the precision is higher for larger s.

Model system:

$$\begin{aligned} y_1' &= -10y_1+\alpha y_2, \\ y_2' &= -\alpha y_1-10y_2, \\ y_3' &= -4y_3, \\ y_4' &= -y_4, \\ y_5' &= -0.5y_5, \\ y_6' &= -0.1y_6, \\ y_i(0) &= 1 \quad (i=1,\ \cdots,\ 6). \end{aligned} \tag{A}$$

Characteristic values of the coefficient matrix of (A):

$$-10\pm i\alpha,\ -4,\ -1,\ -0.5,\ -0.1.$$

Exact solution of (A):

$$\begin{aligned} y_1 &= e^{-10t}(\cos\alpha t+\sin\alpha t), \\ y_2 &= e^{-10t}(\cos\alpha t-\sin\alpha t), \\ y_3 &= e^{-4t}, \\ y_4 &= e^{-t}, \\ y_5 &= e^{-0.5t}, \\ y_6 &= e^{-0.1t}. \end{aligned}$$

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