FINITE DIFFERENCE METHOD FOR A NONLINEAR WAVE EQUATION*

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1. Differential Equation and Difference Equation

For a nonlinear wave equation in viscous flow and elastic mechanics, the existence of its solution has been explored in [1—4]. In this paper, we consider the finite difference method for the initial-boundary value problem of this nonlinear wave equation. We establish prior estimates for the solution of the difference equations on the basis of the prior estimates we prove the convergence and stability of the difference solution and the existence and uniqueness of the classical solution of the differential equation.

We consider the following initial-boundary problem

$$\begin{cases} u_{tt}-u_{xx}=(\sigma(u_x))_x+u_{xxt}-f(u), \quad 0<x<1,\ 0<t\leqslant T, & (1.1)\\ u|_{t=0}=u_0(x),\ u_t|_{t=0}=u_1(x), & (1.2)\\ u(0,\ t)=u(1,\ t)=0, & (1.3)\end{cases}$$

where $\sigma(p)$ and $f(u)$ are given functions, and $u_0(x)$ and $u_1(x)$ are known functions. On the interval [0, 1], the step size of space is h, and the points of the net are $x_0=0$, $x_1=h$, $\cdots$, $x_J=1$. The step size of time is k. We use the following symbols of difference and norm:

$$(u_j^n)_x=\frac{1}{h}(u_{j+1}^n-u_j^n), \qquad (u_j^n)_{\bar{x}}=\frac{1}{h}(u_j^n-u_{j-1}^n),$$

$$(u_j^n)_{\hat{x}}=\frac{1}{2h}(u_{j+1}^n-u_{j-1}^n), \qquad (u_j^n)_{\tilde{x}}=\frac{1}{h}(u_{j+\frac{1}{2}}^n-u_{j-\frac{1}{2}}^n),$$

$$(u_j^n)_{\bar{t}}=\frac{1}{k}(u_j^n-u_j^{n-1}), \qquad (u_j^n,\ v_j^n)=h\sum_{j=0}^{J-1}u_j^n\cdot v_j^n,$$

$$\|u_j^n\|^2=h\sum_{j=0}^{J-1}(u_j^n)^2, \qquad \|u_j^n\|_{L_\infty}=\sup_{0<j<J-1}|u_j^n|.$$

In this paper we use C_i and K_i to denote positive constants.

For the problem (1.1)—(1.3), we give the following implicit scheme

$$\begin{cases} (u_j^{n+1})_{\bar{t}t}-(u_j^{n+1})_{x\bar{x}}=(\sigma((u_j^{n+1})_{\tilde{x}}))_{\tilde{x}}+(u_j^{n+1})_{x\bar{x}\bar{t}}-f(u_j^{n+1}), \\ \qquad\qquad j=1,\ 2,\ \cdots,\ J-1, \quad n=0,\ 1,\ \cdots, & (1.4)\\ u_j^0=u_0(x_j), \quad (u_j^0)_t=u_1(x_j), & (1.5)\\ u_0^n=u_J^n=0, & (1.6)\end{cases}$$

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where u_j^{-1} is solved by the initial condition (1.5). We add quantities u_j^{-2}, u_{-1}^n and u_{-2}^n to solution u_j^n, $1\leqslant j\leqslant J-1$, $n=0, 1, \cdots$. The quantities u_j^{-2}, u_{-1}^n and u_{-2}^n are defined by the following formulas respectively

$$(u_j^0)_{\bar t\bar t}-(u_j^0)_{x\bar x}=(\sigma((u_j^0)_{\bar x}))_{\bar x}+(u_j^0)_{x\bar x\bar t}-f(u_j^0), \tag{1.7}$$

$$(u_0^{n+1})_{\bar t\bar t}-(u_0^{n+1})_{x\bar x}=(\sigma((u_0^{n+1})_{\bar x}))_{\bar x}+(u_0^{n+1})_{x\bar x\bar t}-f(u_0^{n+1}), \tag{1.8}$$

$$(u_0^{n+1})_{\bar t\bar t\bar x}-(u_0^{n+1})_{x\bar x\bar x}=(\sigma((u_0^{n+1})_{\bar x}))_{\bar x\bar x}+(u_0^{n+1})_{x\bar x\bar x\bar t}-[f(u_0^{n+1})]_{\bar x}. \tag{1.9}$$

2. Basic Estimations and Existence of the Solution for the Differential Equation

Lemma 1. *Assume that* (i) $Q(p)=\int_0^p\sigma(r)dr\geqslant 0$, $\sigma(p)\in C^1$, $\sigma'(p)\geqslant 0$, $p\in(-\infty, \infty)$; (ii) $F(u)=\int_0^u f(r)dr\geqslant 0$, $f(u)\in C^1$, $f'(u)\geqslant 0$, $u\in(-\infty, \infty)$; (iii) $u_0(x)\in H_0^1$, $u_1(x)\in L_2$, $\int_0^1 Q(u_0'(x))dx<\infty$, $\int_0^1 F(u_0(x))dx<\infty$. *Then we have the estimations*

$$\|(u_j^n)_{\bar t}\|\leqslant C_1,\quad \|(u_j^n)_x\|\leqslant C_1,\quad \|u_j^n\|\leqslant C_1,$$

$$\|u_j^n\|_{L_\infty}\leqslant C_1,\quad k\sum_{a=1}^{n}\|(u_j^a)_{x\bar t}\|^2\leqslant C_2,$$

$$h\sum_{j=0}^{J-1} Q((u^n_{j+\frac12})_{\bar x})\leqslant C_2,\quad h\sum_{j=0}^{J-1}F(u_j^n)\leqslant C_2,\quad 0\leqslant nk\leqslant T.$$

Proof. Multiplying (1.4) by $(u_j^{n+1})_{\bar t}$ and taking the inner product we have

$$((u_j^{n+1})_{\bar t\bar t}, (u_j^{n+1})_{\bar t})-((u_j^{n+1})_{x\bar x}, (u_j^{n+1})_{\bar t})$$
$$=((\sigma((u_j^{n+1})_{\bar x}))_{\bar x}, (u_j^{n+1})_{\bar t})+((u_j^{n+1})_{x\bar x\bar t}, (u_j^{n+1})_{\bar t})-(f(u_j^{n+1}), (u_j^{n+1})_{\bar t}). \tag{2.1}$$

We deduce the terms of the above formula as follows

$$((u_j^{n+1})_{\bar t\bar t}, (u_j^{n+1})_{\bar t})\geqslant\frac12(\|(u_j^{n+1})_{\bar t}\|^2)_{\bar t},\quad -((u_j^{n+1})_{x\bar x}, (u_j^{n+1})_{\bar t})\geqslant\frac12(\|(u_j^{n+1})_x\|^2)_{\bar t},$$

$$-((\sigma((u_j^{n+1})_{\bar x}))_{\bar x}, (u_j^{n+1})_{\bar t})=(\sigma((u^{n+1}_{j+\frac12})_{\bar x}), (u^{n+1}_{j+\frac12})_{\bar x\bar t})\geqslant h\sum_{j=0}^{J-1}[Q((u^{n+1}_{j+\frac12})_{\bar x})]_{\bar t},$$

$$((u_j^{n+1})_{x\bar x\bar t}, (u_j^{n+1})_{\bar t})=-\|(u_j^{n+1})_{x\bar t}\|^2,$$

$$(f(u_j^{n+1}), (u_j^{n+1})_{\bar t})\geqslant h\sum_{j=0}^{J-1}[F(u_j^{n+1})]_{\bar t}.$$

Thus it follows from (2.1) that

$$\frac12(\|(u_j^{n+1})_{\bar t}\|^2)_{\bar t}+\frac12(\|(u_j^{n+1})_x\|^2)_{\bar t}+h\sum_{j=0}^{J-1}[Q((u^{n+1}_{j+\frac12})_{\bar x})]_{\bar t}$$
$$+\|(u_j^{n+1})_{x\bar t}\|^2+h\sum_{j=0}^{J-1}[F(u_j^{n+1})]_{\bar t}\leqslant 0,$$

$$\|(u_j^{n+1})_{\bar t}\|^2+\|(u_j^{n+1})_x\|^2+2h\sum_{j=0}^{J-1}Q((u^{n+1}_{j+\frac12})_{\bar x})+2k\sum_{a=1}^{n+1}\|(u_j^a)_{x\bar t}\|^2+2h\sum_{j=0}^{J-1}F(u_j^{n+1})$$
$$\leqslant\|(u_j^0)_{\bar t}\|^2+\|(u_j^0)_x\|^2+2h\sum_{j=0}^{J-1}Q((u^0_{j+\frac12})_{\bar x})+2h\sum_{j=0}^{J-1}F(u_j^0). \tag{2.2}$$

From (2.2) and the conditions of the lemma we obtain

$$\|(u_j^{n+1})_{\bar t}\|^2+\|(u_j^{n+1})_x\|^2\leqslant K_1.$$

By Sobolev's embedding theorem, the following inequalities holds

$$\|(u_j^n)_{\bar t}\|\leqslant C_1,\quad \|(u_j^n)_x\|\leqslant C_1,\quad \|u_j^n\|\leqslant C_1,\quad \|u_j^n\|_{L_\infty}\leqslant C_1.$$

In view of the above estimations, (2.2) implies

$$h\sum_{j=0}^{J-1}Q((u^{n+1}_{j+\frac{1}{2}})_{\bar{x}})+k\sum_{\alpha=1}^{n+1}\|(u^{\alpha}_j)_{x\bar{t}}\|^2+h\sum_{j=0}^{J-1}F(u^{n+1}_j)\leqslant K_2,$$

i. e. $$h\sum_{j=0}^{J-1}Q((u^{n+1}_{j+\frac{1}{2}})_{\bar{x}})\leqslant C_2,\quad h\sum_{j=0}^{J-1}F(u^{n+1}_j)\leqslant C_2,\quad k\sum_{\alpha=1}^{n+1}\|(u^{\alpha}_j)_{x\bar{t}}\|^2\leqslant C_2.$$

Lemma 2. *Assume the conditions of Lemma 1 are satisfied. Suppose $u_0(x)\in H^2$ and $u_1(x)\in H^1$. Then we have estimations*

$$\|(u^n_j)_{x\bar{x}}\|\leqslant C_3,\quad \|(u^n_j)_x\|_{L_\infty}\leqslant C_3,\quad 0\leqslant nk\leqslant T.$$

Proof. Multiplying (1.4) by $(u^{n+1}_j)_{x\bar{x}}$ and taking the inner product, we obtain

$$\begin{aligned}((u^{n+1}_j)_{t\bar{t}},\ (u^{n+1}_j)_{x\bar{x}})-\|(u^{n+1}_j)_{x\bar{x}}\|^2=((\sigma((u^{n+1}_j)_{\bar{x}}))_{\bar{x}},\ (u^{n+1}_j)_{x\bar{x}})\\ +((u^{n+1}_j)_{x\bar{x}\bar{t}},\ (u^{n+1}_j)_{x\bar{x}})-(f(u^{n+1}_j),\ (u^{n+1}_j)_{x\bar{x}}).\end{aligned}\tag{2.3}$$

The terms of the above formula are deduced as follows

$$(\sigma((u^{n+1}_j)_{\bar{x}}))_{\bar{x}}=\sigma'(\xi_1)(u^{n+1}_j)_{x\bar{x}},\tag{2.4}$$

where ξ_1 is located between $(u^{n+1}_{j+\frac{1}{2}})_{\bar{x}}$ and $(u^{n+1}_{j-\frac{1}{2}})_{\bar{x}}$.

$$((u^{n+1}_j)_{x\bar{x}\bar{t}},\ (u^{n+1}_j)_{x\bar{x}})\geqslant\frac{1}{2}(\|(u^{n+1}_j)_{x\bar{x}}\|^2)_{\bar{t}},$$

$$((u^{n+1}_j)_{t\bar{t}},\ (u^{n+1}_j)_{x\bar{x}})=-((u^{n+1}_j)_{t\bar{t}x},\ (u^{n+1}_j)_x).$$

Thus it follows from (2.3) that

$$\frac{1}{2}(\|(u^{n+1}_j)_{x\bar{x}}\|^2)_{\bar{t}}+((u^{n+1}_j)_{t\bar{t}x},\ (u^{n+1}_j)_x)\leqslant K_3\|(u^{n+1}_j)_{x\bar{x}}\|^2+K_4.$$

Summing over n from 0 to N, we obtain

$$\begin{aligned}\frac{1}{2}\|(u^{N+1}_j)_{x\bar{x}}\|^2+k\sum_{n=0}^{N}((u^{n+1}_j)_{t\bar{t}x},\ (u^{n+1}_j)_x)\\ \leqslant\frac{1}{2}\|(u^0_j)_{x\bar{x}}\|^2+kK_3\sum_{n=0}^{N}\|(u^{n+1}_j)_{x\bar{x}}\|^2+K_4\cdot T.\end{aligned}\tag{2.5}$$

In addition, as

$$\begin{aligned}k\sum_{n=0}^{N}((u^{n+1}_j)_{t\bar{t}x},\ (u^{n+1}_j)_x)=-((u^{N+1}_j)_{\bar{t}},\ (u^{N+1}_j)_{x\bar{x}})-((u^0_j)_{tx},\ (u^0_j)_x)\\ -k\sum_{n=0}^{N}((u^n_j)_{xt},\ (u^{n+1}_j)_{xt}),\end{aligned}$$

(2.5) implies

$$\begin{aligned}\frac{1}{2}\|(u^{N+1}_j)_{x\bar{x}}\|^2\leqslant\|(u^{N+1}_j)_{\bar{t}}\|^2+\frac{1}{4}\|(u^{N+1}_j)_{x\bar{x}}\|^2+\frac{1}{2}\|(u^0_j)_{tx}\|^2+\frac{1}{2}\|(u^0_j)_x\|^2\\ +\frac{1}{2}k\sum_{n=0}^{N}(\|(u^n_j)_{xt}\|^2+\|(u^{n+1}_j)_{xt}\|^2)+\frac{1}{2}\|(u^0_j)_{x\bar{x}}\|^2+kK_3\sum_{n=0}^{N}\|(u^{n+1}_j)_{x\bar{x}}\|^2+K_4T.\end{aligned}$$

Using the conclusions of Lemma 1 and Gronwall's inequality, we obtain

$$\|(u^n_j)_{x\bar{x}}\|\leqslant C_3.$$

In view of Sobolev's inequality $\|u^n_j\|_{L_\infty}\leqslant K_{15}\|u^n_j\|+K_{16}\|(u^n_j)_{\bar{x}}\|$, we have

$$\|(u^n_j)_x\|_{L_\infty}\leqslant C_3.$$

Lemma 3. *Under the conditions of Lemma 2 we have*

$$\|(u^n_j)_{x\bar{t}}\|\leqslant C_4,\quad 0\leqslant nk\leqslant T.$$

Proof. Multiplying (1.4) by $(u^{n+1}_j)_{\bar{t}}$ and taking the inner product, we obtain

$$\begin{cases} (\varepsilon_j^{n+1})_{\bar{t}t}-(\varepsilon_j^{n+1})_{x\bar{x}}=(\sigma((u(jh,\ (n+1)k))_{\bar{x}}))_{\bar{x}}-(\sigma((u_j^{n+1})_{\bar{x}}))_{\bar{x}} \\ \qquad\qquad +(\varepsilon_j^{n+1})_{x\bar{x}\bar{t}}-f(u(jh,\ (n+1)k))+f(u_j^{n+1})+R_j^{n+1}, & (3.1) \\ \varepsilon_j^0=0, \quad (\varepsilon_j^0)_t=R_j^0, & (3.2) \\ \varepsilon_0^n=\varepsilon_J^n=0, & (3.3) \end{cases}$$

where $|R_j^n|\leqslant K_9(k+h^2),\ n=0,\ 1,\ 2,\ \cdots.$

we have

$$\begin{aligned}
&(\sigma((u(jh,\ (n+1)k))_{\bar{x}}))_{\bar{x}}-(\sigma((u_j^{n+1})_{\bar{x}}))_{\bar{x}} \\
&=\frac{1}{h}\Big[\sigma\Big(\Big(u\Big(\Big(j+\frac{1}{2}\Big)h,\ (n+1)k\Big)\Big)_{\bar{x}}\Big)-\sigma\Big(\Big(u\Big(\Big(j-\frac{1}{2}\Big)h,\ (n+1)k\Big)\Big)_{\bar{x}}\Big)\Big] \\
&\quad-\frac{1}{h}[\sigma((u_{j+\frac{1}{2}}^{n+1})_{\bar{x}})-\sigma((u_{j-\frac{1}{2}}^{n+1})_{\bar{x}})] \\
&=\int_0^1\sigma'\Big(r\Big(u\Big(\Big(j+\frac{1}{2}\Big)h,\ (n+1)k\Big)_{\bar{x}}\Big)+(1-r)\Big(u\Big(\Big(j-\frac{1}{2}\Big)h,\ (n+1)k\Big)\Big)_{\bar{x}}\Big)dr \\
&\quad\times(u(jh,\ (n+1)k))_{x\bar{x}}-\int_0^1\sigma'(r(u_{j+\frac{1}{2}}^{n+1})_{\bar{x}}+(1-r)(u_{j-\frac{1}{2}}^{n+1})_{\bar{x}})dr\cdot(u_j^{n+1})_{x\bar{x}} \\
&=(\varepsilon_j^{n+1})_{x\bar{x}}\cdot\int_0^1\sigma'\Big(r\Big(u\Big(\Big(j+\frac{1}{2}\Big)h,\ (n+1)k\Big)\Big)_{\bar{x}}+(1-r)\Big(u\Big(\Big(j-\frac{1}{2}\Big)h,\ (n+1)k\Big)\Big)_{\bar{x}}\Big)dr \\
&\quad+(u_j^{n+1})_{x\bar{x}}\cdot\int_0^1\Big[\sigma'\Big(r\Big(u\Big(\Big(j+\frac{1}{2}\Big)h,\ (n+1)k\Big)\Big)_{\bar{x}} \\
&\quad+(1-r)\Big(u\Big(\Big(j-\frac{1}{2}\Big)h,\ (n+1)k\Big)\Big)_{\bar{x}}\Big)-\sigma'(r(u_{j+\frac{1}{2}}^{n+1})_{\bar{x}}+(1-r)(u_{j-\frac{1}{2}}^{n+1})_{\bar{x}})\Big]dr \\
&=(\varepsilon_j^{n+1})_{x\bar{x}}\cdot\int_0^1\sigma'\Big(r\Big(u\Big(\Big(j+\frac{1}{2}\Big)h,(n+1)k\Big)\Big)_{\bar{x}}+(1-r)\Big(u\Big(\Big(j-\frac{1}{2}\Big)h,\ (n+1)k\Big)\Big)_{\bar{x}}\Big)dr \\
&\quad+(u_j^{n+1})_{x\bar{x}}\int_0^1\sigma''(\xi_4)[r(\varepsilon_{j+\frac{1}{2}}^{n+1})_{\bar{x}}+(1-r)(\varepsilon_{j-\frac{1}{2}}^{n+1})_{\bar{x}}]dr,
\end{aligned}$$

where ξ_4 is located between $r\Big(u\Big(\Big(j+\frac{1}{2}\Big)h,(n+1)k\Big)\Big)_{\bar{x}}+(1-r)\Big(u\Big(\Big(j-\frac{1}{2}\Big)h,(n+1)k\Big)\Big)_{\bar{x}}$ and $r(u_{j+\frac{1}{2}}^{n+1})_{\bar{x}}+(1-r)(u_{j-\frac{1}{2}}^{n+1})_{\bar{x}}$.

$$|f(u(jh,\ (n+1)k))-f(u_j^{n+1})|\leqslant K_{10}|\varepsilon_j^{n+1}|.$$

Let
$$K_{11}=\max_{|\eta|\leqslant\max(C_6,C_3)}(|\sigma'(\eta)|,\ |\sigma''(\eta)|).$$

Multiplying (3.1) by $(\varepsilon_j^{n+1})_t$ and taking the inner product we obtain

$$\begin{aligned}
&\frac{1}{2}(\|(\varepsilon_j^{n+1})_t\|^2)_{\bar{t}}+\frac{1}{2}(\|(\varepsilon_j^{n+1})_x\|^2)_{\bar{t}}+\|(\varepsilon_j^{n+1})_{x\bar{t}}\|^2 \\
&\leqslant\Big|\Big((\varepsilon_j^{n+1})_{x\bar{x}}\int_0^1\sigma'\Big(r\Big(u\Big(\Big(j+\frac{1}{2}\Big)h,\ (n+1)k\Big)\Big)_{\bar{x}} \\
&\quad+(1-r)\Big(u\Big(\Big(j-\frac{1}{2}\Big)h,\ (n+1)k\Big)\Big)_{\bar{x}}\Big)dr,\ (\varepsilon_j^{n+1})_t\Big)\Big| \\
&\quad+C_7K_{11}(\|(\varepsilon_j^{n+1})_x\|^2+\|(\varepsilon_j^{n+1})_t\|^2) \\
&\quad+\frac{1}{2}K_{10}(\|\varepsilon_j^{n+1}\|^2+\|(\varepsilon_j^{n+1})_t\|^2)+\frac{1}{2}(\|R_j^{n+1}\|^2+\|(\varepsilon_j^{n+1})_t\|^2). \qquad (3.7)
\end{aligned}$$

We make further estimation

$$(\varepsilon_j^{n+1})_{\bar{x}\bar{x}}\int_0^1\sigma'\left(r\left(u\left(\left(j+\frac{1}{2}\right)h,\ (n+1)k\right)\right)_{\bar{x}}+(1-r)\left(u\left(\left(j-\frac{1}{2}\right)h,\ (n+1)k\right)\right)_{\bar{x}}\right)dr$$
$$=\left[(\varepsilon_{j+\frac{1}{2}}^{n+1})_{\bar{x}}\int_0^1\sigma'\left(r\left(u\left(\left(j+\frac{1}{2}\right)h,(n+1)k\right)\right)_{\bar{x}}+(1-r)\left(u\left(\left(j-\frac{1}{2}\right)h,\ (n+1)k\right)\right)_{\bar{x}}dr\right]_{\bar{x}}$$
$$-(\varepsilon_{j-\frac{1}{2}}^{n+1})_{\bar{x}}\cdot\int_0^1\left[\sigma'\left(r\left(u\left(\left(j+\frac{1}{2}\right)h,\ (n+1)k\right)\right)_{\bar{x}}\right.\right.$$
$$\left.+(1-r)\left(u\left(\left(j-\frac{1}{2}\right)h,\ (n+1)k\right)\right)_{\bar{x}}\right]_{\bar{x}}dr,$$
$$\left[\sigma'\left(r\left(u\left(\left(j+\frac{1}{2}\right)h,\ (n+1)k\right)\right)_{\bar{x}}+(1-r)\left(u\left(\left(j-\frac{1}{2}\right)h,\ (n+1)k\right)\right)_{\bar{x}}\right]_{\bar{x}}$$
$$=\sigma''(\xi_5)\cdot\left[r\left(u\left(\left(j+\frac{1}{2}\right)h,\ (n+1)k\right)\right)_{\bar{x}\bar{x}}+(1-r)\left(u\left(\left(j-\frac{1}{2}\right)h,\ (n+1)k\right)\right)_{\bar{x}\bar{x}}\right],$$

where ξ_5 is located between $r\left(u\left(\left(j+\frac{1}{2}\right)h,\ (n+1)k\right)\right)_{\bar{x}}+(1-r)\left(u\left(\left(j-\frac{1}{2}\right)h,(n+1)k\right)\right)_{\bar{x}}$ and $r\left(u\left(\left(j-\frac{1}{2}\right),\ (n+1)k\right)\right)_{\bar{x}}+(1-r)\left(u\left(\left(j-\frac{3}{2}\right)h,\ (n+1)k\right)\right)_{\bar{x}}$.

Thus it follows from (3.7) that

$$\frac{1}{2}(\|(\varepsilon_j^{n+1})_{\bar{t}}\|^2)_{\bar{t}}+\frac{1}{2}(\|(\varepsilon_j^{n+1})_x\|^2)_{\bar{t}}+\|(\varepsilon_j^{n+1})_{x\bar{t}}\|^2$$
$$\leqslant\left|\left((\varepsilon_{j+\frac{1}{2}}^{n+1})_{\bar{x}}\int_0^1\sigma'\left(r\left(u\left(\left(j+\frac{1}{2}\right)h,\ (n+1)k\right)\right)_{\bar{x}}\right.\right.\right.$$
$$\left.\left.\left.+(1-r)\left(u\left(\left(j-\frac{1}{2}\right)h,\ (n+1)k\right)\right)_{\bar{x}}\right)dr,\ (\varepsilon_j^{n+1})_{x\bar{t}}\right)\right|$$
$$+C_7K_{11}|((\varepsilon_{j-\frac{1}{2}}^{n+1})_{\bar{x}},\ (\varepsilon_j^{n+1})_{\bar{t}})|+C_7K_{11}(\|(\varepsilon_j^{n+1})_x\|^2+\|(\varepsilon_j^{n+1})_{\bar{t}}\|^2)$$
$$+\frac{1}{2}K_{10}(\|\varepsilon_j^{n+1}\|^2+\|(\varepsilon_j^{n+1})_{\bar{t}}\|^2)+\frac{1}{2}(\|R_j^{n+1}\|^2+\|(\varepsilon_j^{n+1})_{\bar{t}}\|^2)$$
$$\leqslant\frac{1}{2}(K_{11}^2\|(\varepsilon_j^{n+1})_x\|^2+\|(\varepsilon_j^{n+1})_{x\bar{t}}\|^2)+2C_7K_{11}(\|(\varepsilon_j^{n+1})_x\|^2+\|(\varepsilon_j^{n+1})_{\bar{t}}\|^2)$$
$$+\frac{1}{2}K_{10}(\|\varepsilon_j^{n+1}\|^2+\|(\varepsilon_j^{n+1})_{\bar{t}}\|^2)+\frac{1}{2}(\|R_j^{n+1}\|^2+\|(\varepsilon_j^{n+1})_{\bar{t}}\|^2),$$

i.e.

$$\|(\varepsilon_j^{n+1})_{\bar{t}}\|^2+\|(\varepsilon_j^{n+1})_x\|^2\leqslant\|(\varepsilon_j^0)_{\bar{t}}\|^2+\|(\varepsilon_j^0)_x\|^2+k\sum_{\alpha=1}^{n+1}\|R_j^\alpha\|^2$$
$$+K_{12}k\sum_{\alpha=1}^{n+1}(\|(\varepsilon_j^\alpha)_{\bar{t}}\|^2+\|(\varepsilon_j^\alpha)_x\|^2+\|\varepsilon_j^\alpha\|^2).$$

Using Gronwall's inequality and the estimations for R_j^n, we have

$$\|(\varepsilon_j^{n+1})_{\bar{t}}\|^2+\|(\varepsilon_j^{n+1})_x\|^2\leqslant K_{13}(k+h^2).$$

From Sobolev's embedding theorem we obtain the conclusion of the theorem.

Theorem 4. *Under the conditions of Theorem* 1 *the difference scheme* (1.4)—(1.6) *is stable.*

Proof. Suppose there exist solutions of the difference equations u_j^n and v_j^n, which satisfy the difference equations (1.4) and boundary condition (1.6). But their initial conditions are different. Let $\varepsilon_j^n=u_j^n-v_j^n$; then we obtain the equations satisfied by the error

$$(\varepsilon_j^{n+1})_{t\bar{t}}-(\varepsilon_j^{n+1})_{x\bar{x}}=(\sigma((u_j^{n+1})_{\bar{x}}))_x-(\sigma((v_j^{n+1})_{\bar{x}}))_x+(\varepsilon_j^{n+1})_{x\bar{x}\bar{t}}-f(u_j^{n+1})+f(v_j^{n+1}),$$

$$\varepsilon_j^0 = u_j^0 - v_j^0, \quad (\varepsilon_j^0)_{\bar{t}} = (u_j^0)_{\bar{t}} - (v_j^0)_{\bar{t}},$$
$$\varepsilon_0^n = \varepsilon_J^n = 0.$$

Similarly to the proof of Theorem 3, it is easy to prove

$$\|(\varepsilon_j^{n+1})_{\bar{t}}\|^2 + \|(\varepsilon_j^{n+1})_x\|^2 \leqslant K_{14} \cdot (\|(\varepsilon_j^0)_{\bar{t}}\|^2 + \|(\varepsilon_j^0)_x\|^2),$$

i.e. the difference scheme (1.4)—(1.6) is stable.

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