THE GENERALIZED PATCH TEST FOR ZIENKIEWICZ'S TRIANGLES* 1)

SHI ZHONG-CI (石钟慈)

(China University of Science and Technology, Hefei, China)

Abstract

It is proved that Zienkiewicz's triangles for plate bending problems pass Stummel's generalized patch test——a necessary and sufficient condition for convergence of nonconforming finite elements——for mesh (a) generated by three sets of parallel lines, but do not pass it when "union jack" mesh (b) or when another mesh (c) is used. In the latter two cases the approximations are divergent.

1. Introduction

It is well known that Zienkiewicz's triangles[1] for plate bending problems are nonconforming, since the gradients of the shape functions are discontinuous at interelement boundaries. Concerning the convergence of this element, numerical experiments in [1, 2] have shown that mesh (a) of Figure 1, generated by three sets of parallel lines (called for brevity the condition of parallel lines), guarantees convergence, whereas mesh (b) of Figure 2 composed of "union jack" figures does not give convergence. In order to explain why Zienkiewicz's triangles were convergent in one configuration but not in others, Irons-Razzaque created the patch test[3] and showed that Zienkiewicz's triangles pass the test under the condition of parallel lines, but do not pass it for the union jack configuration.

Later on, Lascaux and Lesaint[4] gave a mathematical proof of the convergence of Zienkiewicz's triangles under the condition of parallel lines and derived corresponding error estimates for the plate problem. More recently, Stummel[5, 6] pointed out that the patch test of Irons is neither necessary nor sufficient for convergence of nonconforming elements, and proposed a generalized patch test instead, which does indeed give both necessary and sufficient conditions for convergence. Stummel proved in [5] that various nonconforming elements pass this generalized patch test; however, Zienkiewicz's triangles were not analysed in that paper.

Since passing the patch test is no longer necessary for convergence, it is not proved yet whether mesh (b) and mesh (c) of Figure 3, that do not pass Irons patch test, diverge or not. Concerning mesh (c), the authors in [1] state: "the convergence is most unlikely, and this case has not been investigated numerically".

We shall prove in this paper that:

(i) Zienkiewicz's triangles pass the generalized patch test under the condition of

* Received January 16, 1984.

1) This work was done while the author was visiting the University of Frankfurt, Federal Republic of Germany, on a grant by the Alexander von Humboldt Foundation.

parallel lines;

(ii) mesh (b) and mesh (c) do not pass the generalized patch test and, thus, do not converge.

According to Stummel's theory[7], the validity of the generalized patch test, together with the approximability condition and strong continuity condition at interelement boundaries (the latter two conditions are satisfied by Zienkiewicz's triangles for arbitrary decompositions), provide the preconditions for the validity of a generalized Rellich compactness theorem. As a consequence thereof, very general stability and convergence theorems are valid about approximations of general coercive elliptic variational equations and eigenvalue problems with variable, not necessarily smooth coefficients.

2. Zienkiewicz's Triangles under the Condition of Parallel Lines

We consider a triangulation $\mathscr{K}_h$ of a given polyhedral domain $G\subset\mathbb{R}^2$ with finite elements K. Let $h(K)=$diameter of K, $h=\max\limits_{K\in\mathscr{K}_h}\{h(K)\}$, $\rho(K)=$ the greatest diameter of the circles inscribed in K. We assume that the triangulation $\mathscr{K}_h$ is regular[8], that is, there exists a constant σ independent of h such that

$$h(K)\leqslant\sigma\rho(K),\quad K\in\mathscr{K}_h, \tag{1}$$

when the greatest diameter h approaches zero.

Given a triangle K with vertices $p_i=(x_i,\ y_i)$, $1\leqslant i\leqslant 3$, we let λ_i denote the area coordinates relative to the vertices p_i, $\triangle$ the area of K, and

$$\begin{aligned}&\xi_1=x_{23}=x_2-x_3,\quad \xi_2=x_{31}=x_3-x_1,\quad \xi_3=x_{12}=x_1-x_2,\\&\eta_1=y_{23}=y_2-y_3,\quad \eta_2=y_{31}=y_3-y_1,\quad \eta_3=y_{12}=y_1-y_2.\end{aligned} \tag{2}$$

Zienkiewicz's triangles are thus defined as follows (see [9, p. 187]):

(i) Nodal parameters are the function values and the values of the gradients at the vertices of K (In case of Dirichlet boundary conditions nodal parameters are zero at the vertices on the boundary.);

(ii) The space $\mathscr{P}(K)$ of the shape functions w is a space of polynomials of third degree having the following form:

$$\begin{aligned}w(p)=a_1\lambda_1+a_2\lambda_2+a_3\lambda_3&+a_4\left(\lambda_1^2\lambda_2+\frac{1}{2}\lambda_1\lambda_2\lambda_3\right)+a_5\left(\lambda_2^2\lambda_1+\frac{1}{2}\lambda_1\lambda_2\lambda_3\right)\\&+a_6\left(\lambda_2^2\lambda_3+\frac{1}{2}\lambda_1\lambda_2\lambda_3\right)+a_7\left(\lambda_3^2\lambda_2+\frac{1}{2}\lambda_1\lambda_2\lambda_3\right)\\&+a_8\left(\lambda_3^2\lambda_1+\frac{1}{2}\lambda_1\lambda_2\lambda_3\right)+a_9\left(\lambda_1^2\lambda_3+\frac{1}{2}\lambda_1\lambda_2\lambda_3\right).\end{aligned} \tag{3}$$

The unique polynomial in $\mathscr{P}(K)$, determined by its nodal parameters described above, is

$$w(p)=\sum_{i=1}^{3}[\varphi_i w(p_i)+\psi_i w_x(p_i)+\rho_i w_y(p_i)], \tag{4}$$

where

$$\varphi_i=\lambda_i^2(3-2\lambda_i)+2\lambda_1\lambda_2\lambda_3, \tag{5}$$

$$\psi_i=\xi_{i+1}\left(\lambda_i^2\lambda_{i+2}+\frac{1}{2}\lambda_1\lambda_2\lambda_3\right)-\xi_{i+2}\left(\lambda_i^2\lambda_{i+1}+\frac{1}{2}\lambda_1\lambda_2\lambda_3\right), \tag{6}$$

$$\rho_i=\eta_{i+1}\left(\lambda_i^2\lambda_{i+2}+\frac{1}{2}\lambda_1\lambda_2\lambda_3\right)-\eta_{i+2}\left(\lambda_i^2\lambda_{i+1}+\frac{1}{2}\lambda_1\lambda_2\lambda_3\right), \tag{7}$$

with $\lambda_{i+j}=\lambda_k,\ \xi_{i+j}=\xi_k,\ \eta_{i+j}=\eta_k,\quad i+j\equiv k \pmod 3.$

It is also shown in [8] that the space $\mathscr{P}(K)$ includes the constant curvature states, i.e.

$$\mathscr{P}_2\subset\mathscr{P}(K)\subset\mathscr{P}_3,$$

where $\mathscr{P}_r$ denotes the space of all polynomials of at most r-th degree.

Now let V_h be the finite element spaces of functions defined on $\bar{G}$, whose restrictions to each element K are the polynomials w in $\mathscr{P}(K)$. For a fourth order problem, the validity of the generalized patch test for the spaces V_h consists in showing that for every bounded sequence $w_h\in V_h$ and for $h\to 0$, the following relations

$$\text{(i)}\qquad T_r(\psi,\ w)=\sum_K\int_{\partial K}\psi w_h N_r\,ds\to 0,\quad r=1,\ 2, \tag{8}$$

$$\text{(ii)}\qquad T_{rk}(\psi,\ w)=\sum_K\int_{\partial K}\psi\,\frac{\partial w_h}{\partial x_k}N_r\,ds\to 0,\quad r,\ k=1,\ 2 \tag{9}$$

hold for all test functions $\psi\in C_0^\infty(G)$ ($\psi\in C_0^\infty(\mathbb{R}^2)$ in case of Dirichlet boundary conditions), where N_r are the components of the unit vector in the outward normal direction on the boundary of the element K.

Theorem 1. *Under the condition of parallel lines the finite element spaces V_h pass the generalized patch test.*

Proof. (i) By definition of V_h, functions $w_h\in V_h$ are continuous together with their partial derivatives of first order at the vertices of the triangles. Since a polynomial of third degree in one variable on an interval F is uniquely determined by its function values and first derivatives at the endpoints of F, it follows immediately that the w_h are continuous on $\bar{G}$, so that $T_r(\psi,\ w)$ represents a telescoping sum in which the terms cancel pairwise. Therefore (8) follows.

(ii) In proving (9), we use a similar technique as in [4]. Consider now the case $r=1$, $k=2$. (The other cases can be dealt with in an analogous way.) For brevity, we omit the suffix h in w_h and write $\frac{\partial w}{\partial x_1}=w_x$, $\frac{\partial w}{\partial x_2}=w_y$, $N_1=N_x$, $N_2=N_y$, so that

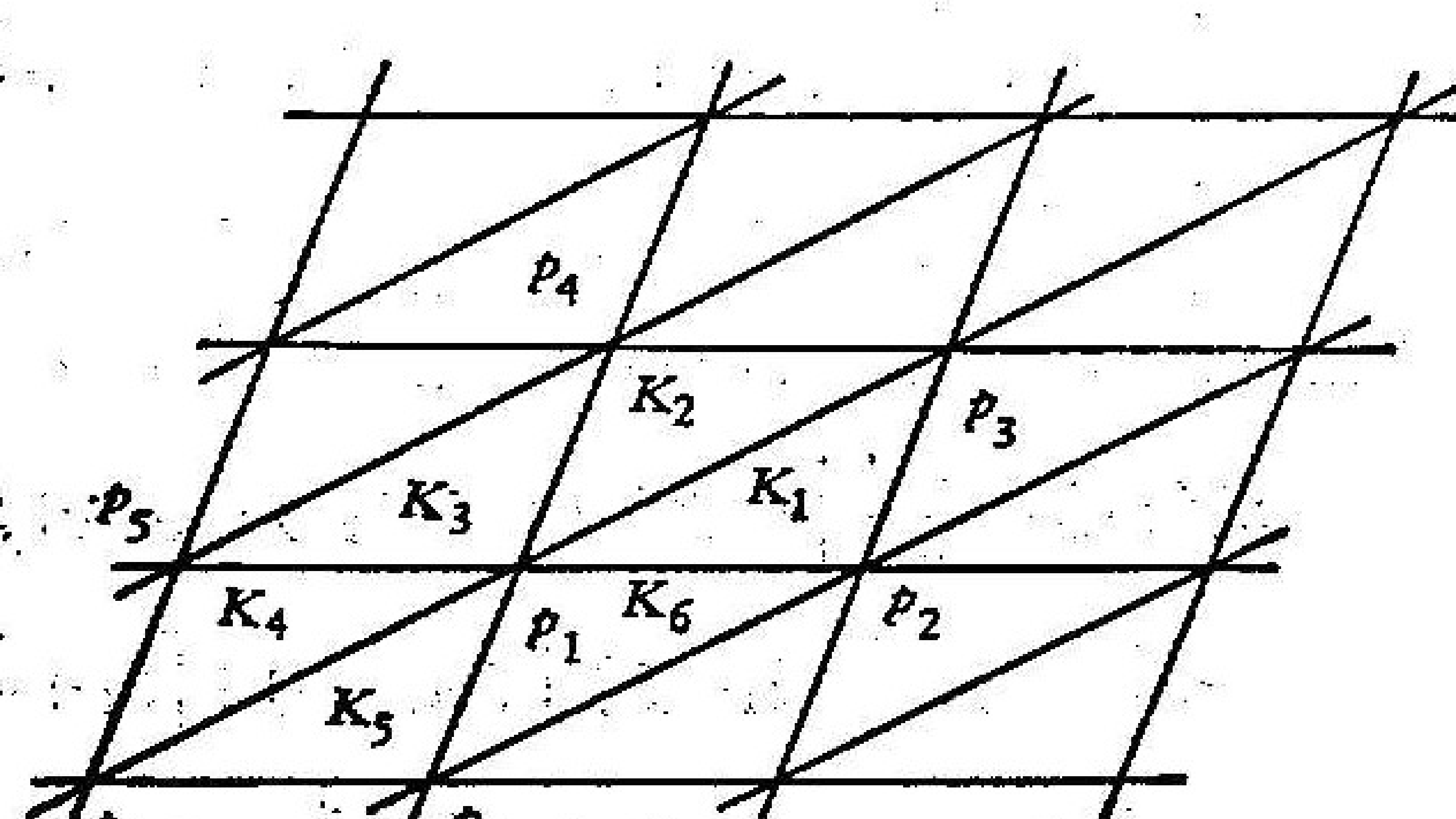
$$T_{12}(\psi,\ w)=\sum_K\int_{\partial K}\psi w_y N_x\,ds. \tag{10}$$

Because of the condition of parallel lines, the edges of all triangles of $\mathscr{K}_h$ are parallel to three directions $a=p_1p_2$, $b=p_2p_3$, $c=p_3p_1$. Hence $T_{12}(\psi,\ w)$ can be written in the form

$$T_{12}=T^a+T^b+T^c, \tag{11}$$

where T^a denotes the sum of the integrals over all edges parallel to a, with T^b and T^c defined similarly.

Now, let us consider T^c. Let $K_1=\triangle p_1p_2p_3$ and $K_2=\triangle p_1p_3p_4$ be two



Mesh (a)

Fig. 1

adjacent triangles with common edge p_3p_1. The case that $p_3p_1\subset\partial G$ will be discussed below. Then the integrals over p_3p_1 are

$$T^c_{p_3p_1}=\int_{p_3p_1}\psi w^1_y N^1_x\,ds+\int_{p_3p_1}\psi w^2_y N^2_x ds=N^1_x\int_{p_3p_1}\psi(w^1_y-w^2_y)ds, \tag{12}$$

where w^1, w^2 denote, respectively, the restrictions of w to K_1 and K_2, with N^1_x, N^2_x the components of the unit outward normal vectors on p_3p_1 relative to K_1 and K_2. Since $w^1_y-w^2_y$ is a polynomial of second degree in one variable on p_3p_1 vanishing at the endpoints, we have

$$w^1_y-w^2_y=4\left(1-\frac{t}{l_{31}}\right)\frac{t}{l_{31}}(w^1_y-w^2_y)(p_{31}),\quad 0\leqslant t\leqslant l_{31}, \tag{13}$$

where p_{31} denotes the midpoint of p_3p_1, l_{31} is the length of p_3p_1, and t is an abscissa along p_3p_1. Therefore,

$$T^c_{p_3p_1}=4y_{13}(w^1_y-w^2_y)(p_{31})\int_0^1\psi_{31}(s)s(1-s)ds \tag{14}$$

with

$$\psi_{31}(s)=\psi(p_3+s(p_1-p_3)). \tag{15}$$

Recalling now the definition of w in (3), one can verify that

$$\frac{\partial w^1}{\partial\lambda_1}(p_{31})=\frac{3}{2}w(p_1)+\frac{1}{2}\xi_2 w_x(p_1)-\frac{1}{4}\xi_2 w_x(p_3)+\frac{1}{2}\eta_2 w_y(p_1)-\frac{1}{4}\eta_2 w_y(p_3),$$

$$\begin{aligned}\frac{\partial w^1}{\partial\lambda_2}(p_{31})=&\frac{1}{2}(w(p_1)+w(p_2)+w(p_3))+\frac{1}{8}(\xi_2-3\xi_3)w_x(p_1)+\frac{1}{8}(\xi_3-\xi_1)w_x(p_2)\\&+\frac{1}{8}(3\xi_1-\xi_2)w_x(p_3)+\frac{1}{8}(\eta_2-3\eta_3)w_y(p_1)+\frac{1}{8}(\eta_3-\eta_1)w_y(p_2)\\&+\frac{1}{8}(3\eta_1-\eta_2)w_y(p_3),\end{aligned}$$

$$\frac{\partial w^1}{\partial\lambda_3}(p_{31})=\frac{3}{2}w(p_3)+\frac{1}{4}\xi_2 w_x(p_1)-\frac{1}{2}\xi_2 w_x(p_3)+\frac{1}{4}\eta_2 w_y(p_1)-\frac{1}{2}\eta_2 w_y(p_3),$$

with the result that

$$\begin{aligned}w^1_y(p_{31})=&\frac{-1}{2\Delta}\left(\xi_1\frac{\partial w^1}{\partial\lambda_1}+\xi_2\frac{\partial w^1}{\partial\lambda_2}+\xi_3\frac{\partial w^1}{\partial\lambda_3}\right)(p_{31})=\frac{-1}{4\Delta}\Big\{(3\xi_1+\xi_2)w(p_1)\\&+\xi_2 w(p_2)+(3\xi_3+\xi_2)w(p_3)+\left[\xi_1+\frac{1}{4}(\xi_2-\xi_3)\right]\xi_2 w_x(p_1)+\frac{1}{4}(\xi_3-\xi_1)\xi_2 w_x(p_2)\\&+\left[-\xi_3+\frac{1}{4}(\xi_1-\xi_2)\right]\xi_2 w_x(p_3)+\left[\eta_2\xi_1+\frac{1}{4}(\eta_2-3\eta_3)\xi_2+\frac{1}{2}\eta_2\xi_3\right]w_y(p_1)\\&+\frac{1}{4}(\eta_3-\eta_1)\xi_2 w_y(p_2)+\left[-\frac{1}{2}\eta_2\xi_1+\frac{1}{4}(3\eta_1-\eta_2)\xi_2-\eta_2\xi_3\right]w_y(p_3)\Big\}.\end{aligned} \tag{16}$$

Using the condition of parallel lines, we obtain a similar equality for the triangle K_2:

$$\begin{aligned}w^2_y(p_{31})=&\frac{-1}{4\Delta}\Big\{-(3\xi_3+\xi_2)w(p_1)-(3\xi_1+\xi_2)w(p_3)-\xi_2 w(p_4)\\&+\left[-\xi_3+\frac{1}{4}(\xi_1-\xi_2)\right]\xi_2 w_x(p_1)+\left[\xi_1+\frac{1}{4}(\xi_2-\xi_3)\right]\xi_2 w_x(p_3)\\&+\frac{1}{4}(\xi_3-\xi_1)\xi_2 w_x(p_4)+\left[-\frac{1}{2}\eta_2\xi_1+\frac{1}{4}(3\eta_1-\eta_2)\xi_2-\eta_2\xi_3\right]w_y(p_1)\\&+\left[\eta_2\xi_1+\frac{1}{4}(\eta_2-3\eta_3)\xi_2+\frac{1}{2}\eta_2\xi_3\right]w_y(p_3)+\frac{1}{4}(\eta_3-\eta_1)\xi_2 w_y(p_4)\Big\}.\end{aligned} \tag{17}$$

Hence,

$$(w_y^1-w_y^2)(p_{31})=\frac{x_{31}}{4\Delta}\Big\{w(p_1)-w(p_2)+w(p_3)-w(p_4)+\frac{1}{4}x_{31}(w_x(p_1)-w_x(p_3))$$
$$+\frac{1}{4}(x_{12}-x_{23})(w_x(p_4)-w_x(p_2))+\frac{1}{4}y_{31}(w_y(p_1)-w_y(p_3))$$
$$+\frac{1}{4}(y_{12}-y_{23})(w_y(p_4)-w_y(p_2))\Big\}=\frac{x_{31}}{8\Delta}\{(w_{12}-w_{43})+(w_{14}-w_{23})\}, \tag{18}$$

where

$$w_{ij}=w(p_i)-w(p_j)-\frac{x_{ij}}{2}(w_x(p_i)+w_x(p_j))-\frac{y_{ij}}{2}(w_y(p_i)+w_y(p_j)). \tag{19}$$

By substituting (19) into (14), we get

$$T^c_{p_3p_1}=M_{31}\{(w_{12}-w_{43})+(w_{14}-w_{23})\}, \tag{20}$$

where

$$M_{ij}=-\frac{x_{ij}y_{ij}}{2\Delta}\int_0^1\psi_{ij}(s)s(1-s)ds. \tag{21}$$

Equation (20) shows that the integrals over p_3p_1 consist of two parts, one relating to terms on p_1p_2 and p_4p_3 parallel to a——that is $w_{12}-w_{43}$——and the other relating to terms on p_1p_4 and p_2p_3 parallel to b——that is $w_{14}-w_{23}$. The common multiple M_{31} depends on the diagonal p_3p_1 of the parallelogram $p_1p_2p_3p_4$.

For $p_3p_1\subset\partial G$, $K_1\in\mathscr{K}_h$, we have, for the case of Dirichlet boundary conditions,

$$T^c_{p_3p_1}=M_{31}(w_{12}-w_{23}). \tag{22}$$

Otherwise, since $\psi\in C_0^\infty(G)$, we conclude

$$T^c_{p_3p_1}=0. \tag{23}$$

Thus, if we obtain T^c by adding the integrals over all edges parallel to c, we derive the following decomposition:

$$T^c=\sum_{s/\!/a}G_1(s)+\sum_{s/\!/b}G_2(s), \tag{24}$$

where $G_1(s)$, $G_2(s)$ are the terms associated with all the edges parallel to a and b, respectively. For example,

$$G_1(p_1p_2)=(M_{31}-M_{27})w_{12}, \tag{25}$$
$$G_2(p_1p_4)=(M_{31}-M_{45})w_{14}. \tag{26}$$

On the other hand, Taylor expansion yields

$$w_{ij}=-\frac{1}{12}(x_{ij}^3w_{xxx}+3x_{ij}^2y_{ij}w_{xxy}+3x_{ij}y_{ij}^2w_{xyy}+y_{ij}^3w_{yyy}).$$

All the third derivatives are constant since w is a polynomial of third degree on K with $p_ip_j\in K$, so that

$$|w_{ij}|\leqslant C_1h^3|w|_{3,K}. \tag{27}$$

Using the inverse property, we get

$$|w_{ij}|\leqslant C_2h|w|_{2,K}, \tag{28}$$

where all C_i will denote generic constants independent of h.

Furthermore, it may be seen that

$$\left|\int_0^1(\psi_{31}-\psi_{27})s(1-s)\,ds\right|\leqslant C_3|\psi|_{1,K_1\cup K_2}, \tag{29}$$

$$\left|\int_0^1 (\psi_{31}-\psi_{45})s(1-s)\,ds\right| \leqslant C_4 |\psi|_{1,K_1\cup K_2}; \tag{30}$$

and, by the assumption of a regular triangulation, it follows that

$$\left|\frac{x_{ij}y_{ij}}{2\Delta}\right| \leqslant C_5. \tag{31}$$

Combining inequalities (29), (30) and (31), we get

$$|G_1(p_1p_2)| \leqslant C_6 h |\psi|_{1,K_1\cup K_4} |w|_{2,K_1},$$
$$|G_2(p_1p_4)| \leqslant C_7 h |\psi|_{1,K_2\cup K_3} |w|_{2,K_1},$$

and so

$$|T^c| \leqslant C_8 h |\psi|_1 |w|_{2,h}, \quad |w|^2_{2,h} = \sum_K |w|^2_{2,K}. \tag{32}$$

The other two terms T^a and T^b can be treated in the same way. Thus we have

$$|T_{12}(\psi, w)| \leqslant C_9 h |\psi|_1 |w|_{2,h}, \quad w \in V_h, \tag{33}$$

for all test functions $\psi \in C_0^\infty(G)$ ($\psi \in C_0^\infty(\mathbb{R}^2)$ in case of Dirichlet boundary conditions). This means that the test is satisfied.

Under the condition of parallel lines, we have shown that Zienkiewicz's triangles pass the generalized patch test.

3

Theorem 2. *Mesh (b) and mesh (c) do not pass the generalized patch test.*

We shall show, that for meshes (b) and (c), there exist sequences of trial functions w_h, $\bar{w}_h \in V_h$, respectively, and a test function $\psi \in C_0^\infty(G)$ ($\psi \in C_0^\infty(\mathbb{R}^2)$ in case of Dirichlet boundary conditions) such that the test (9) does not hold. Hence Zienkiewicz's triangles for mesh (b) and mesh (c) are divergent.

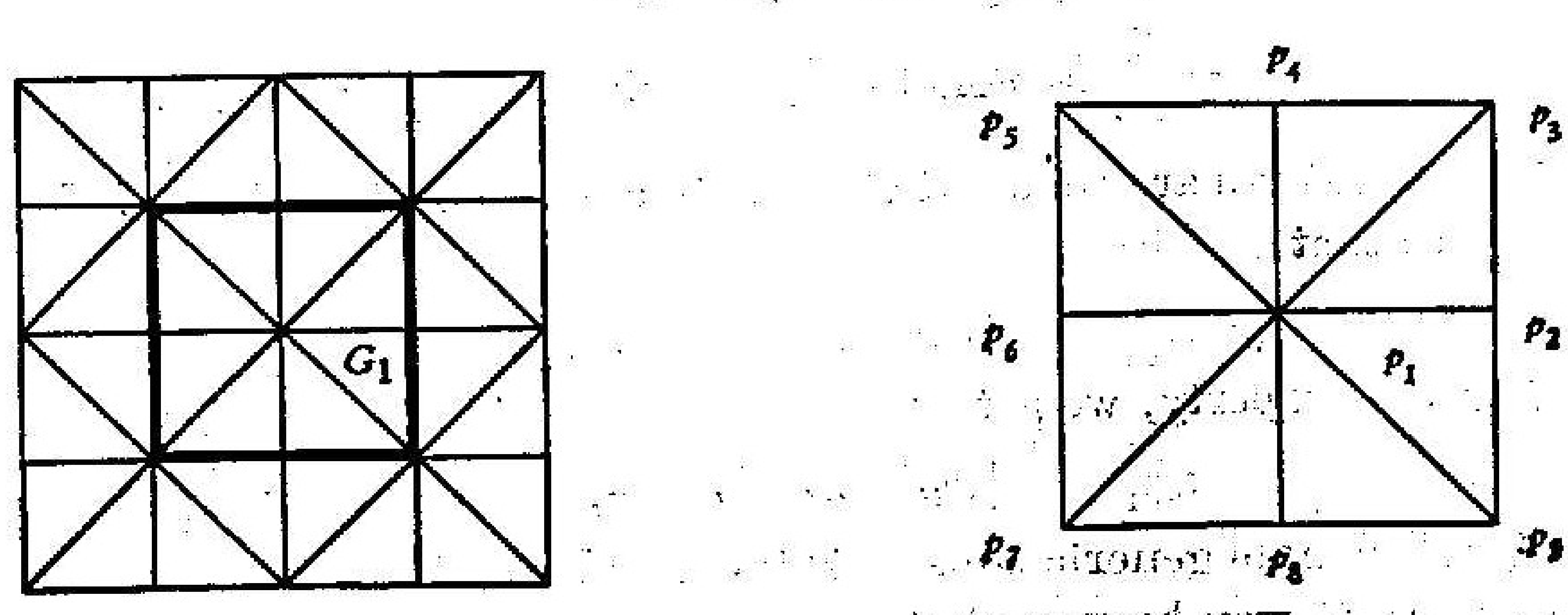
Proof. (i) Mesh (b).

Given a unit square $G=(0, 1)\times(0, 1)$, we consider a triangulation of G by right isosceles triangles K with mesh pattern (b) (Fig. 2). The mesh sizes are

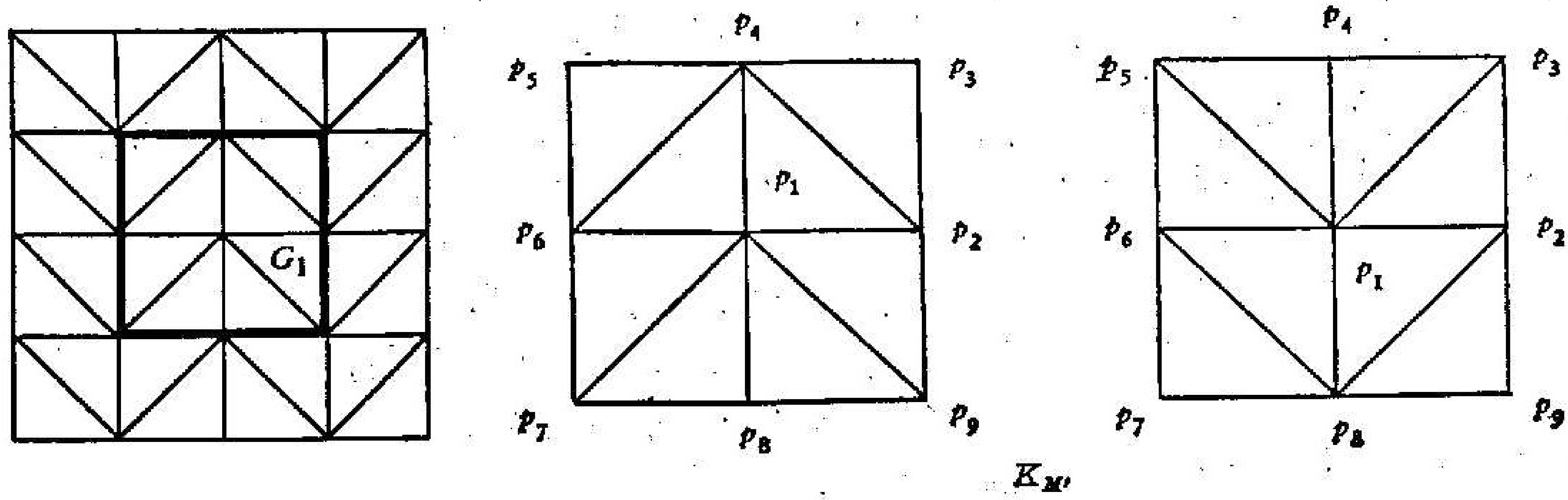
$$h_n = \frac{1}{2^n}, \quad n=2, \cdots$$

in both x and y directions. Then choose in G the fixed subdomain

$$G_1 = \left[\frac{1}{4}, \frac{3}{4}\right] \times \left[\frac{1}{4}, \frac{3}{4}\right].$$



Mesh (b)
Fig. 2



Mesh (c)

Fig. 3

There are $N_1=2^{2n-4}$ (for $n\geqslant 2$) squares of side-length $2h_n$ in G_1 having the mesh pattern K_M of Figure 2 with midpoints p_1. The eight mesh points on the outer boundaries of the K_M are denoted by p_i, $2\leqslant i\leqslant 9$.

Now let us define a special sequence of trial functions $w\in V_h$ as follows: for $p\in K_M$, let $w(p_1)=1$, $w'_x(p_1)=w'_y(p_1)=0$,

$$w(p_i)=w'_x(p_i)=w'_y(p_i)=0, \quad 2\leqslant i\leqslant 9, \tag{34}$$

and for $p\in G\backslash G_1$, let $w(p)=w_x(p)=w_y(p)=0$.

We further define a test function $\psi\in C_0^\infty(G)$ such that

$$\psi\equiv 1 \quad \text{on } G_1. \tag{35}$$

In case of Dirichlet boundary conditions, we, in addition, choose

$$\psi\equiv 0 \quad \text{on } \mathbb{R}^2\backslash G.$$

By virtue of the special choices of w and ψ,

$$\sum_{K\in K_M}\int_{\partial K}\psi\frac{\partial w}{\partial x}N_x\,ds=-\frac{4}{3}, \quad \|w\|^2_{0,K_M}=C_0^2h_n^2, \quad C_0^2=\frac{242}{315}. \tag{36}$$

By applying the inverse property, we have

$$|w|_{1,K_M}\leqslant\frac{C_1}{h_n}\|w\|_{0,K_M}=C_0C_1, \quad |w|_{2,K_M}\leqslant\frac{C_2}{h_n}|w|_{1,K_M}\leqslant\frac{C_0C_1C_2}{h_n},$$

and so

$$\|w\|_{2,K_M}\leqslant\frac{C_3}{h_n}(1+O(h_n)).$$

Hence

$$T_{11}(\psi, w)=\sum_K\int_{\partial K}\psi\frac{\partial w}{\partial x}N_x\,ds=-\frac{4}{3}N_1=-\frac{1}{12}\frac{1}{h_n^2},$$

$$\|w\|_{2,h}\leqslant\frac{C_4}{h_n^2}(1+O(h_n)),$$

$$\frac{|T_{11}(\psi, w)|}{\|w\|_{2,h}}\geqslant C_5(1+O(h_n)), \quad C_5\neq 0. \tag{37}$$

This shows that for the bounded sequence $\frac{w}{\|w\|_{2,h}}\in V_h$ defined by (34), and for the test function ψ defined by (35), the bilinear form T_{11} does not tend to zero as $h\to 0$. Zienkiewicz's triangles for mesh (b) fail to pass the generalized patch test.

(ii) Mesh (c).

We refer the reader to Figure 3 for the triangulation of $G=(0, 1)\times(0, 1)$ and its subdomain

$$G_1=\left[\frac{1}{4},\ \frac{3}{4}\right]\times\left[\frac{1}{4},\ \frac{3}{4}\right],$$

as well as the squares $K_{M'}$ in G_1. There are $N_1=2^{2n-4}$ of $K_{M'}$ in G_1 with mesh points p_1 as its center and the $p_i(2\leqslant i\leqslant 9)$ located on the boundaries of the $K_{M'}$.

Let $\bar{w}\in V_h$ be defined as follows:

$$\bar{w}(p_1)=\bar{w}_x(p_1)=0,\quad \bar{w}_y(p_1)=1,$$
$$\bar{w}(p_i)=\bar{w}_x(p_i)=\bar{w}_y(p_i)=0,\quad 2\leqslant i\leqslant 9,\ p\in K_{M'}, \tag{38}$$

and

$$\bar{w}(p)=\bar{w}_x(p)=\bar{w}_y(p)=0,\quad p\in G\backslash G_1.$$

The test function ψ is defined as in (35).

After some algebraic manipulations, it is found that

$$\sum_{K\in K_{M'}}\int_{\partial K}\psi\,\frac{\partial\bar{w}}{\partial x}N_x ds=-\frac{2}{3}h_n,\quad \|\bar{w}\|^2_{0,K_{M'}}=C_0^2h_n^4,\ C_0^2=\frac{9}{560}. \tag{39}$$

Using the inverse property we have

$$\|\bar{w}\|_{2,K_{M'}}\leqslant C_1(1+O(h_n)),$$

so that

$$T_{11}(\psi,\ \bar{w})=-\frac{2}{3}h_nN_1=-\frac{1}{24}\frac{1}{h_n},$$

$$\|\bar{w}\|_{2,h}\leqslant\frac{C_2}{h_n}(1+O(h_n)),$$

$$\frac{|T_{11}(\psi,\ \bar{w})|}{\|\bar{w}\|_{2,h}}\geqslant C_3(1+O(h_n)),\quad C_3\neq 0. \tag{40}$$

Therefore Zienkiewicz's triangles for mesh (c) fail to pass the generalized patch test also.

It is worth remarking that the sequence w defined by (34) gives $T_{r,k}(\psi,\ w)=0$, $r,\ k=1,\ 2$, for mesh (c) and, therefore, can not be used for the proof in this case.

The author is most grateful to Prof. F. Stummel for suggesting this investigation and helpful discussions.

References

[1] G. P. Bazeley, et al., Triangular elements in plate bending-conforming and nonconforming solutions, Conf. on Matrix Methods in Struct, Mechs., AFFDL-TR-66-80, 1965, 547—576.

[2] B. M. Irons, et al., Comments on the paper: Theoretical foundations of the finite element method. *Int. J. Solids Struct*, 6, 1970, 695—697.

[3] B. M. Irons, A. Razzaque, Experience with the patch test for convergence of finite elements, Proc. Symp. on the Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Operators (Baltimore 1972), A. K. Aziz, ed., Academic Press, New York, 1972, 557—587.

[4] P. Lascaux, P. Lesaint, Some nonconforming finite elements for the plate bending problem, *RAIRO Anal. Numér.*, 9, 1975, 9—53.

[5] F. Stummel, The limitations of the patch test, *Int. J. Numer. Meth. Eng.*, 15, 1980, 177—188.

[6] F. Stummel, The generalized patch test, *SIAM J. Numer. Anal.*, 16, 1979, 449—471.

[7] F. Stummel, Basic compactness properties of nonconforming and hybrid finite element spaces, *RAIRO Anal. Numér.*, 4, 1980, 81—115.

[8] P. G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.

[9] O. C. Zienkiewicz, The Finite Element Method in Engineering Science, McGraw-Hill, London, 1971.