EXISTENCE OF RATIONAL INTERPOLATION FUNCTION AND AN OPEN PROBLEM OF P. TURÁN*

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Abstract

An existence theorem of rational interpolation function for the sufficient condition has correctly been stated by Macon-Dupree in [2], but some arguments in their proof are not true. In this paper:(i) A related theorem for both the sufficient and necessary condition is asserted and proved by a new and rigorous approach, namely by introducing the notion of (m/n) quasi-rational interpolant of a given function. (ii) With use of these results thus obtained an open problem proposed by P. Turán in [4] is completely solved.

§ 1. Introduction

Let $f(x)$ be a bounded real-valued function defined on an interval $[a, b]$, let m and n be non-negative integers, and let $x_i\in[a, b]$ $(i=0, 1, \cdots, m+n)$ be distinct points. The problem of rational interpolation is that of finding a rational function $R(x)=R\in\boldsymbol{R}(m, n)$ satisfying

$$R(x_i)=y_i \qquad (y_i=f(x_i);\ i=0, 1, \cdots, m+n), \tag{1.1}$$

where $\boldsymbol{R}(m, n)=\{R\colon R=N/D,\ N\in H_m,\ D\in H_n\backslash\{0\}\}$,

herein H_k denotes the class of all polynomials of degree at most k.

As we know, while the problem of polynomial interpolation is constantly solvable, the solution of the problem (1.1) does not always exist ([1, p. 2], [2, p. 754]). In order to get its possible solution, one may consider the linearized interpolation problem satisfying, instead of condition (1.1), the following linear equations:

$$N(x_i)-y_iD(x_i)=0, \qquad i=0, 1, \cdots, m+n. \tag{1.1a}$$

Now, this system of $m+n+1$ homogeneous equations in $m+n+2$ unknowns has always nontrivial solutions. However, the two problems (1.1) and (1.1a) are not equivalent. From the following known theorems one may then find their conditioned connections.

Theorem 1.1([1, p. 5], [2, p. 754], [5, p. 54]). *There exists a rational function $R\in\boldsymbol{R}(m, n)$ satisfying condition* (1.1) *if and only if the pair $\tilde{N}$ and $\tilde{D}$, obtained by dividing out all common factors in any nontrivial solution $N\in H_m$ and $D\in H_n$ of* (1.1a), *remains to be a solution of* (1.1a).

Another more practical and useful theorem may be stated in a convenient form by introducing notations for the matrices

$$C(\mu, \nu)=\begin{pmatrix} 1 & x_0 & \cdots & x_0^\mu & y_0 & y_0x_0 & \cdots & y_0x_0^\nu \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_{m+n} & \cdots & x_{m+n}^\mu & y_{m+n} & y_{m+n}x_{m+n} & \cdots & y_{m+n}x_{m+n}^\nu \end{pmatrix} \tag{1.2}$$

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and $C_i(\mu, \nu)$ which denotes the matrix obtained by deleting i-th row of $C(\mu, \nu)$. We note immediately that the rational function $R(x)=\sum_{i=0}^{\mu} a_i x^i / \sum_{i=0}^{\nu} b_i x^i$, which corresponds to the nontrivial solution $\xi=(a_0 \cdots a_\mu - b_0 \cdots - b_\nu)^T$ of the equation $C(\mu, \nu)\ \xi=0$, satisfies (1.1a). Thereby, we have

Theorem 1.2([1, p. 14], [2, p. 758]). *If the rank of $C_i\ (m-1, n-1)$ is constant for $i=0, 1, \cdots, m+n$, there exists a rational function $R \in \boldsymbol{R}\ (m, n)$ satisfying the interpolation condition* (1.1).

Nevertheless, we should remark in passing that Theorem 1.2 is a true one, but within its proof, as given in the quoted references, some assertion concerning the magnitude of the rank of the related matrices can not hold. A simple counter-example, such as, given $m=n=2$, $N(x)=D(x)=x(\alpha x+\beta)$, $\beta \neq 0$, with the interpolating points (0, 0), (1, 1), (2, 1), (3, 1) and (4, 1), will do the illustration.

In the next section we make further effort to scrutinize the existence problem of rational interpolation functions. A new, rigorous proof is given for the existence Theorem 1.2 in an extended sense, namely, the condition therein given is not only sufficient but also necessary (see Theorems 2.2—2.4).

Using these concerned results as background we have solved an open problem in the field of approximation theory proposed by P. Turán in 1974 ([4, p. 79], Problem LXXXII). The problem is as follows:

Let m, n be given. For $m+n+1$ variable knots $x_0, x_1, \cdots, x_{m+n}$, what is the maximal number $M=M(m, n)$ such that, at least M of the relations (1.1) can be satisfied for any choice of y_i?

§ 2. Existence of Rational Interpolation Functions

Let $\quad \boldsymbol{R}_0(m, n)=\{R: R \in \boldsymbol{R}(m, n)$ satisfying (1.1a)$\}$.

From (1.1a), we easily get the following

Lemma 2.1. *Let $N/D \in \boldsymbol{R}_0(m, n)$. Then $N=0$ if and only if at least $m+1$ of the y_i's vanish.*

On account of Lemma 2.1, it follows that, in case $k (k \geqslant m+1)$ of the y_i's vanish, the problem (1.1) is solvable if and only if all of the y_i's vanish. Hence, unless specially remarked, we always assume, throughout this section, that no more than m of the y_i's can be zeros. Thus, for any $N/D \in \boldsymbol{R}_0(m, n)$, we have $N \neq 0$. Now, if we define

$$\begin{aligned} m^* &= \min\{\partial(N):\ N/D \in \boldsymbol{R}_0(m, n)\}, \\ n^* &= \min\{\partial(D):\ N/D \in \boldsymbol{R}_0(m, n)\}, \end{aligned} \tag{2.1}$$

then $m^* \geqslant 0$, $n^* \geqslant 0$. Here $\partial(P)$ denotes the degree of polynomial P, and we define $\partial(0) = -1$.

Lemma 2.2(cf. [3, p. 295]). *Let N/D and $N_1/D_1 \in \boldsymbol{R}_0(m, n)$. Then $ND_1=N_1D$.*

Lemma 2.3. *For m^* and n^* defined by* (2.1), *there exists the unique (without counting the common constant factor in numerator and denominator) $R^*=N^*/D^* \in \boldsymbol{R}_0(m, n)$ such that*

$$\partial(N^*)=m^*, \qquad \partial(D^*)=n^*;$$

and that any $R=N/D \in \boldsymbol{R}_0(m, n)$ can be reduced into R^* by dividing out some common *factor*.

Proof. Suppose there exist both N^*/D^* and $N_1/D_1\in \boldsymbol{R}_0(m, n)$ such that, $\partial(N^*)=m^*$, $\partial(D^*)\geqslant n^*$ and $\partial(N_1)\geqslant m^*$, $\partial(D_1)=n^*$. From Lemma 2.2, we have $\partial(N^*)+\partial(D_1)=\partial(N_1)+\partial(D^*)$. Consequently, $\partial(D^*)=n^*$. So the existence of R^* is proved.

Let N/D be any element in $\boldsymbol{R}_0(m, n)$. Then $\partial(N)\geqslant m^*$, $\partial(D)\geqslant n^*$. We may write

$$\begin{aligned} N&=q_1N^*+r_1, \qquad \partial(r_1)<m^*,\\ D&=q_2D^*+r_2, \qquad \partial(r_2)<n^*. \end{aligned} \tag{2.2}$$

By Lemma 2.2, we can get $q_1=q_2=q$. Furthermore, using (2.2) we have

$$r_1(x_i)-y_ir_2(x_i)=0, \qquad i=0, 1, \cdots, m+n.$$

It follows, from the definition of m^* and n^*, that $r_1=r_2=0$. This implies the second conclusion of the lemma. In particular, if $\partial(N)=m^*$, $\partial(D)=n^*$, then q is a constant. Hence we get the uniqueness of R^*.

For ease of discrimination, the rational function R^* in Lemma 2.3 is hereafter called the (m/n) quasi-rational interpolant of f over the knots x_i $(i=0, 1, \cdots, m+n)$. Lemma 2.3 indicates that the quasi-rational interpolant of f always exists and is unique. Meanwhile, we have

Lemma 2.4. *Let $R^*=N^*/D^*\in \boldsymbol{R}_0(m, n)$ be the (m/n) quasi-rational interpolant of f. Then for any x_i $(0\leqslant i\leqslant m+n)$,*

$$R^*(x_i)=y_i$$

if and only if $D^*(x_i)\neq 0$.

Proof. If $D^*(x_i)\neq 0$, it follows from (1.1a) that $R^*(x_i)=y_i$. Conversely, if $D^*(x_i)=0$, then $N^*(x_i)=0$. Let $N^*=(x-x_i)N$ and $D^*=(x-x_i)D$. It turns out $R^*(x_i)\neq y_i$. For, if $R^*(x_i)=y_i$, we get $N/D\in \boldsymbol{R}_0(m, n)$. This contradicts the definition of R^*.

By Lemma 2.4, we can immediately get the following theorem, which is similar to Theorem 1.1 but can be stated in a clear and definitive manner.

Theorem 2.1. *Let $R^*=N^*/D^*\in \boldsymbol{R}_0(m, n)$ be the (m/n) quasi-rational interpolant of f. There exists a solution of the rational interpolation problem (1.1) if and only if $D^*(x_i)\neq 0$ for $i=0, 1, \cdots, m+n$.*

Lemma 2.5. *Suppose m_1 and n_1 are integers possessing following properties:*

(i) $0\leqslant m_1\leqslant m$, $0\leqslant n_1\leqslant n$;

(ii) *the rank of $C(m_1, n_1)$ is less than m_1+n_1+2;*

(iii) *the matrices $C(m_1-1, n_1)$ and $C(m_1, n_1-1)$ have the same rank m_1+n_1+1.*

Then

$$m_1=m^*, \qquad n_1=n^*,$$

where m^* and n^* are those defined by (2.1). Conversely, the concerned m^* and n^* possess the same properties as m_1 and n_1, and of course the rational function, corresponding to the nontrivial solution of the equation

$$C(m^*, n^*)\xi=0 \tag{2.3}$$

is the (m/n) quasi-rational interpolant of f.

Proof. If m_1 and n_1 have properties (i), (ii) and (iii), then the equation

$$C(m_1, n_1)\xi=0$$

has the unique nontrivial solution (without counting a constant factor). It is evident that the rational function $R=N/D$ corresponding to this solution is in $\boldsymbol{R}_0(m, n)$, where $N\in H_{m_1}$ and $D\in H_{n_1}$. Hence we have by Lemma 2.3 that $N=qN^*$ and $D=qD^*$. Now if $\partial(q)\geqslant 1$, $N^*/D^*\in \boldsymbol{R}_0(m, n)$ implies that the equation

$$C(m_1-\partial(q),\ n_1-\partial(q))\xi=0$$

has nontrivial solutions. This contradicts (iii). Consequently, $\partial(q)=0$. It means that $m_1=m^*$, $n_1=n^*$.

Conversely, for m^*, n^* defined by (2.1), property (i) holds obviously. By Lemma 2.3, the equation (2.3) has the nontrivial solution corresponding to the (m/n) quasi-rational interpolant of f. Therefore (ii) must be valid. From the minimality of m^* and n^*, property (iii) follows. So the lemma is completely proved.

It is worthy to remark here that from the minimality of m^* and n^*, we can also get directly

Lemma 2.6. *The matrix $C(m^*-1, n)$ has rank m^*+n+1 and the matrix $C(m, n^*-1)$ has rank $m+n^*+1$.*

Now, we are ready to establish the following theorems.

Theorem 2.2. *Let $R^*=N^*/D^*\in \boldsymbol{R}_0(m, n)$ be the (m/n) quasi-rational interpolant of f. Then for any x_i $(0\leqslant i\leqslant m+n)$,*

$$R^*(x_i)=y_i$$

if and only if $\boldsymbol{r}\{C_i(m^*-1,\ n^*-1)\}=m^*+n^*$, where $\boldsymbol{r}\{A\}$ denotes the rank of matrix A.

Proof. (a) Suppose $\boldsymbol{r}\{C_i(m^*-1,\ n^*-1)\}=m^*+n^*$ and $R^*(x_i)\neq y_i$. Then we have by Lemma 2.4 that $D^*(x_i)=0$, so $N^*(x_i)=0$. Let us put

$$N^*(x)=(x-x_i)N_1(x),\quad D^*(x)=(x-x_i)D_1(x).$$

Then for $0\leqslant j(\neq i)\leqslant m+n$,

$$N_1(x_j)-y_jD_1(x_j)=[N^*(x_j)-y_jD^*(x_j)]/(x_j-x_i)=0.$$

It implies that the equation

$$C_i(m^*-1,\ n^*-1)\xi=0 \tag{2.4}$$

has nontrivial solutions. Therefore, $\boldsymbol{r}\{C_i(m^*-1,\ n^*-1)\}<m^*+n^*$, a contradiction. Thus, $R^*(x_i)=y_i$.

Amid the above sufficiency proof, we should mention in passing that it must be $N_1(x_i)-y_iD_1(x_i)\neq 0$. Otherwise, $N_1/D_1\in \boldsymbol{R}_0(m, n)$, and this contradicts the definition of R^*. This fact means that, if there exist common roots in the numerator and the denominator of the (m/n) quasi-rational interpolant, each of them must be simple.

(b) We turn to prove the necessity part of the theorem. Suppose $\boldsymbol{r}\{C_i(m^*-1,\ n^*-1)\}<m^*+n^*$. Then equation (2.4) must have a nontrivial solution. Let its corresponding rational function be $R_1=N_1/D_1$. Then

$$N_1(x_j)-y_jD_1(x_j)=0,\qquad j=0,\ \cdots,\ i-1,\ i+1,\ \cdots,\ m+n.$$

Define

$$N(x)=(x-x_i)N_1(x),\quad D(x)=(x-x_i)D_1(x).$$

Then $N/D\in \boldsymbol{R}_0(m, n)$ and $\partial(N)\leqslant m^*$, $\partial(D)\leqslant n^*$. From the minimality of m^* and n^*, it follows that N/D must be the (m/n) quasi-rational interpolant. However,

$D(x_i)=0$. This causes, by Lemma 2.4, a contradiction. So the proof is completed.

It follows at once from Theorem 2.2 the following

Theorem 2.3. *There exists a solution of the rational interpolation problem* (1.1) *if and only if for all i $(0\leqslant i\leqslant m+n)$,*

$$r\{C_i(m^*-1,\ n^*-1)\}=m^*+n^*.$$

Now, in what follows, we are going to prove from Theorem 1.2 the extended theorem of existence, where the extension means that the condition specified in the former is not only sufficient but also necessary.

Theorem 2.4.[1)] *There exists a solution of rational interpolation problem* (1.1) *if and only if the rank of $C_i(m-1,\ n-1)$ is constant for $i=0,\ 1,\ \cdots,\ m+n$.*

Proof. (a) By Lemma 2.5, we have $r\{C(m^*-1,\ n^*-1)\}=m^*+n^*$. So the matrix $C(m^*-1,\ n^*-1)$ can be augmented with some other columns from $C(m-1,\ n-1)$ in such a way that it contains only the maximal independent system of the columns of the latter. Let $C'(m^*-1,\ n^*-1)$ denote the matrix thus augmented. Then

$$r\{C'(m^*-1,\ n^*-1)\}=r\{C(m-1,\ n-1)\}.$$

Suppose

$$r\{C_i(m-1,\ n-1)\}=k,\qquad i=0,\ 1,\ \cdots,\ m+n.$$

Of course, we have presently

$$k=r\{C(m-1,\ n-1)\}=r\{C'(m^*-1,\ n^*-1)\}.$$

Now, if there exists no solution of the problem (1.1), it follows from Theorem 2.2 that there must be some index i_0 $(0\leqslant i_0\leqslant m+n)$ such that the rank of $C_{i_0}(m^*-1,\ n^*-1)$ is less than m^*+n^*. Therefore,

$$r\{C'_{i_0}(m^*-1,\ n^*-1)\}<k. \tag{2.5}$$

However, from the definition of $C'(m^*-1,\ n^*-1)$, we have that every column of $C(m-1,\ n-1)$ can be linearly expressed by the columns of $C'(m^*-1,\ n^*-1)$. Naturally, each column of $C_i(m-1,\ n-1)$ can also be linearly expressed by the columns of $C'_i(m^*-1,\ n^*-1)$. This means that for $i=0,\ 1,\ \cdots,\ m+n$,

$$r\{C'_i(m^*-1,\ n^*-1)\}=k.$$

This contradicts (2.5). Thus the interpolation problem (1.1) must have a solution.

(b) We assume that there exists a solution of the problem (1.1). We shall prove in turn that, for $i=0,\ 1,\ \cdots,\ m+n$, the rank of $C_i(m-1,\ n-1)$ is constant. Let $r\{C(m-1,\ n-1)\}=k$ and C' be a matrix consisting of a maximal independent system of the columns of the matrix $C(m-1,\ n-1)$. Then

$$r\{C'_i\}=r\{C_i(m-1,\ n-1)\}\leqslant k,\qquad i=0,\ 1,\ \cdots,\ m+n.$$

Therefore, it needs only to prove that for $i=0,\ 1,\ \cdots,\ m+n$, $r\{C'_i\}=k$. Now, if there is i_0 $(0\leqslant i_0\leqslant m+n)$ such that $r\{C'_{i_0}\}<k$, then the rational function $R_1=N_1/D_1$ $(N_1\in H_{m-1},\ D_1\in H_{n-1})$, corresponding to a nontrivial solution of the equation $C'_{i_0}\xi=0$, satisfies

$$\begin{aligned}&N_1(x_i)-y_iD_1(x_i)=0,\quad i=0,\ 1,\ \cdots,\ m+n;\ i\neq i_0,\\&N_1(x_{i_0})-y_{i_0}D_1(x_{i_0})\neq 0.\end{aligned} \tag{2.6}$$

1) **Remark.** It is not difficult to verify directly from (1.2) that in case more than m of the y_i's vanish, Theorem 2.4 still holds.

The inequality (2.6) can easily be obtained from the fact that $\boldsymbol{r}\{C'\}=k$. Let

$$N(x)=(x-x_{i_0})N_1(x), \quad D(x)=(x-x_{i_0})D_1(x).$$

Then $R=N/D\in \boldsymbol{R}_0(m, n)$. By Lemma 2.3, R can be reduced into the (m/n) quasi-rational interpolant by dividing out some common factor. The inequality (2.6) implies that the factor $(x-x_{i_0})$ can not be divided out. Thus $D^*(x_{i_0})=0$. By Lemma 2.4, this yields a contradiction. Therefore, for $i=0, 1, \cdots, m+n$, $\boldsymbol{r}\{C_i(m-1, n-1)\}=k$. This completes the proof of the theorem, which is the main theorem of existence in our paper.

As a corollary of Theorem 2.4 in the case $\boldsymbol{r}\{C(m, n)\}=m+n+1$, we shall speak about it at length. This case was treated in [1, p. 11] and [2, p. 755] as the main theorem. Therefrom Theorem 1.2 was offered to be its generalization. However, as we have remarked in § 1, there are unsatisfactory points for Theorem 1.2. Here, we have rather started from a different approach. The following corollary may be simply deduced from Theorem 2.4.

Corollary. Suppose the rank of $C(m, n)$ is $m+n+1$. Then there exists a solution of the rational interpolation problem (1.1) if and only if all the matrices $C_i(m-1, n-1)$, $i=0, 1, \cdots, m+n$, are nonsingular.

Proof. If $\boldsymbol{r}\{C(m, n)\}=m+n+1$, the solution space S of equations (1.1a) must be one-dimensional. Hence both $m^*<m$ and $n^*<n$ can not hold simultaneously. Otherwise the rational functions obtained by multiplying the numerator and the denominator of the (m/n) quasi-rational interpolant by a linear factor are still in $\boldsymbol{R}_0(m, n)$. This leads to a contradiction for S.

Without loss of generality, we may assume that $m^*=m$. This means that the $(m+1)$-th column of the matrix $C(m, n)$ can be linearly expressed by others. Therefore,

$$\boldsymbol{r}\{C(m, n)\}=\boldsymbol{r}\{C(m-1, n)\}=m+n+1,$$

and of course

$$\boldsymbol{r}\{C(m-1, n-1)\}=m+n.$$

Consequently, at least one of the matrices $C_i(m-1, n-1)$ $(i=0, 1, \cdots, m+n)$ has rank $m+n$. By Theorem 2.4, the corollary holds.

With reference to the full rank of $C(m, n)$, it is noteworthy to indicate some points of interest.

(i) The condition "$\boldsymbol{r}\{C(m, n)\}=m+n+1$" in our corollary is equivalent to the condition "$\beta(x, y)\neq 0$" of the theorem in the quoted references [1] and [2], where

$$\beta(x, y)=\det\begin{pmatrix} & & C(m, n) & & \\ 1 & x\cdots x^m & y & yx\cdots yx^n \end{pmatrix}.$$

(ii) From Lemma 2.6, we have

$$\boldsymbol{r}\{C(m, n)\}\geqslant \max\{m^*+n+1, m+n^*+1\}. \tag{2.7}$$

In fact, if we notice that for any polynomial $h\in H_{\min(m-m^*, n-n^*)}$, $hN^*/hD^*\in \boldsymbol{R}_0(m, n)$, then we can prove that

$$\boldsymbol{r}\{C(m, n)\}=\max\{m^*+n+1, m+n^*+1\}. \tag{2.8}$$

It follows that the condition "$\boldsymbol{r}\{C(m, n)\}=m+n+1$" is equivalent to $m^*=m$ or $n^*=n$.

§ 3. A Problem of P. Turán

In 1974, P. Turán proposed 89 open problems in the field of approximation theory[4]. Some of them have been completely or partially solved in recent years. Literatures dealing with some of these problems are listed in [6]. In this paper we only consider Problem LXXXII (see, §1), which was proposed when Professor Turán studied the convergence of rational interpolation functions.

First, we answer this problem in $\boldsymbol{R}_0(m, n)$.

Theorem 3.1. *For any $R\in \boldsymbol{R}_0(m, n)$, at least $m+1$ of the relations (1.1) can be satisfied; and for arbitrarily given distinct points x_i $(0\leqslant i\leqslant m+n)$, there exist y_i $(0\leqslant i\leqslant m+n)$ such that, for any $R\in \boldsymbol{R}_0(m, n)$, at most $m+1$ of the relations (1.1) can be satisfied, i.e., $M=m+1$ in $\boldsymbol{R}_0(m, n)$.*

Proof. For any $R=N/D\in \boldsymbol{R}_0(m, n)$, if $R(x_i)\neq y_i$, we must have $D(x_i)=0$. Therefore, at least $m+n+1-\partial(D)$ of the relations (1.1) are valid. It follows $M\geqslant m+1$.

On the other hand, taking $y_0=y_1=\cdots=y_m=0$, $y_{m+1}=\cdots=y_{m+n}=1$, then we have, by Lemma 2.1, that for any $R=N/D\in \boldsymbol{R}_0(m, n)$, $N=0$. Hence, only $m+1$ of the relations (1.1) hold, i.e., $M\leqslant m+1$.

Now, we proceed to the solution of Turán's problem in $\boldsymbol{R}(m, n)$. First of all, we establish the following lemmas.

Lemma 3.1. *Let (x_i, y_i), $i=1, 2, \cdots, n+k$, $k\geqslant 1$, be $n+k$ points with x_i distinct. There exists $P_n(x)\in H_n$ such that*

$$P_n(x_i)=y_i,\ i=1, 2, \cdots, n; \qquad P_n(x_{n+i})\neq y_{n+i},\ i=1, 2, \cdots, k.$$

Proof. Choose polynomials $P_{n-1}(x)$, $Q_{n-1}(x)\in H_{n-1}$ such that

$$P_{n-1}(x_i)=y_i, \quad Q_{n-1}(x_i)=x_i^n, \qquad i=1, 2, \cdots, n.$$

Define

$$P_n(\lambda, x)=P_{n-1}(x)+\lambda[x^n-Q_{n-1}(x)].$$

Obviously, for any λ, $P_n(\lambda, x_i)=y_i\,(i=1, 2, \cdots, n)$, and for any $x\neq x_i$ $(i=1, 2, \cdots, n)$, $x^n-Q_{n-1}(x)\neq 0$ (otherwise $Q_{n-1}(x)\equiv x^n$). Hence, for $i=1, 2, \cdots, k$, $x_{n+i}^n-Q_{n-1}(x_{n+i})\neq 0$ and there exists λ_0 such that

$$\lambda_0[x_{n+i}^n-Q_{n-1}(x_{n+i})]\neq y_{n+i}-P_{n-1}(x_{n+i}).$$

It follows that $P_n(\lambda_0, x_{n+i})\neq y_{n+i}$ for $i=1, 2, \cdots, k$. The polynomial $P_n(\lambda_0, x)$ is what we want.

Lemma 3.2. *Suppose the rank of $C(m, n)$ is less than $m+n+1$. Then we can properly change one point of (x_i, y_i) $(i=0, 1, \cdots, m+n)$ such that the rank of $C(m, n)$ is increased by at least one.*

Proof. Let $\boldsymbol{r}\{C(m, n)\}=r$. From the property of Vandermonde determinant, $m+1\leqslant r<m+n+1$. For definiteness, we may assume that the first r rows of $C(m, n)$ are linearly independent and may assume the same for the first r columns. This means that the matrix

$$E=\begin{pmatrix} 1 & x_0 & \cdots & x_0^m & y_0 & y_0x_0 & \cdots & y_0x_0^{n'} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_{r-1} & \cdots & x_{r-1}^m & y_{r-1} & y_{r-1}x_{r-1} & \cdots & y_{r-1}x_{r-1}^{n'} \end{pmatrix}$$

is nonsingular, where $n'=r-(m+2)$ (in case of $r=m+1$, by the definition $\partial(0)=-1$, the matrix E has only first $m+1$ columns).

Consider the determinant

$$\det\begin{pmatrix} & E & & \begin{matrix} y_0 x_0^{n'+1} \\ \vdots \\ y_{r-1} x_{r-1}^{n'+1} \end{matrix} \\ 1\ x\ \cdots\ x^m & y\ yx\ \cdots\ yx^{n'} & & yx^{n'+1} \end{pmatrix} = \alpha_m(x) + y\beta_{n'+1}(x),$$

where $\alpha_m \in H_m$, $\beta_{n'+1} \in H_{n'+1}$. Since the matrix E is nonsingular, $\beta_{n'+1} \neq 0$. Therefore, there exists a point (x^*, y^*) such that

$$\alpha_m(x^*) + y^* \beta_{n'+1}(x^*) \neq 0.$$

Consequently, replacing the point (x_r, y_r) by (x^*, y^*), we get that the rank of $C(m, n)$ is increased by at least one.

Lemma 3.3. *If the rank of $C(m, n)$ is $m+n+1$, then we can change at most one point of (x_i, y_i) $(i=0, 1, \cdots, m+n)$ such that all the matrices $C_i(m-1, n-1)$, $i=0, 1, \cdots, m+n$, are nonsingular.*

Proof. If every matrix $C_i(m-1, n-1)$ is nonsingular, the lemma is obviously valid. From the proof of the corollary in §2, it follows that at least one of the matrices $C_i(m-1, n-1)$ is nonsingular. Without loss of generality, we may assume that this matrix is $C_0(m-1, n-1)$. Since any two matrices $C_i(m-1, n-1)$ and $C_j(m-1, n-1)$ $(0 \leqslant i \neq j \leqslant m+n)$ have $m+n-1$ common rows, the matrices $C_i(m-1, n-1)$ $(1 \leqslant i \leqslant m+n)$ have rank at least $m+n-1$. We shall show below that the point (x_0, y_0) may be so changed that every matrix $C_i(m-1, n-1)$ $(1 \leqslant i \leqslant m+n)$ can be made nonsingular. Since the matrices $C_i(m-1, n-1)$ $(i=1, 2, \cdots, m+n)$ are all related to (x_0, y_0) and we may regard (x_0, y_0) as a variable point (x, y), the determinant of each matrix $C_i(m-1, n-1)$ $(1 \leqslant i \leqslant m+n)$, similar to the proof of Lemma 3.2, can be expressed as follows:

$$\det C_i(m-1, n-1) = \alpha_{m-1}^{(i)}(x) + y\beta_{n-1}^{(i)}(x), \qquad i=1, 2, \cdots, m+n.$$

Because all the rows of $C_i(m-1, n-1)$ $(1 \leqslant i \leqslant m+n)$, except the row corresponding to (x, y), are included in $C_0(m-1, n-1)$, the relations $\alpha_{m-1}^{(i)}=0$ and $\beta_{n-1}^{(i)}=0$ can not be valid simultaneously. Now, let the set of zeros be defined by

$$Z=\{x:\ \alpha_{m-1}^{(i)}(x)=0 \quad \text{but} \quad \alpha_{m-1}^{(i)} \neq 0, \qquad \text{or}$$
$$\beta_{n-1}^{(i)}(x)=0 \quad \text{but} \quad \beta_{n-1}^{(i)} \neq 0, \qquad i=1, 2, \cdots, m+n\}.$$

Then Z is a finite set. We may arbitrarily choose one point $x^* \overline{\in} Z \cup \{x_i: 0 \leqslant i \leqslant m+n\}$ and put

$$\alpha_{m-1}^{(i)}(x^*) = a_i, \quad \beta_{n-1}^{(i)}(x^*) = b_i.$$

Then $|a_i| + |b_i| \neq 0$ for $i=1, 2, \cdots, m+n$. Hence the system of inequalities

$$a_i + b_i y \neq 0, \quad i=1, 2, \cdots, m+n$$

is obviously consistent. Let y^* be one of its solutions. Thus, with use of (x^*, y^*) instead of (x_0, y_0), every $C_i(m-1, n-1)$ is made to be nonsingular. This concludes the proof of the lemma.

Theorem 3.2. *Let $x_i (i=0, 1, \cdots, m+n)$ be arbitrary and distinct points. Then* *for* y_i $(i=0, 1, \cdots, m+n)$ *arbitrarily given, there exists a rational function* $R\in\boldsymbol{R}(m, n)$ *such that at least* $m+E\left[\frac{n}{2}\right]+1$ *of relations* (1.1) *can be satisfied; furthermore, there exist* y_i $(i=0, 1, \cdots, m+n)$ *such that, for any* $R\in\boldsymbol{R}(m, n)$, *at most* $m+E\left[\frac{n}{2}\right]+1$ *of relations* (1.1) *can be satisfied, i.e.,*

$$M=m+E\left[\frac{n}{2}\right]+1,$$

where $E[\lambda]$ *denotes the maximal integer not greater than* λ.

Proof. (a) We shall show that $M\leqslant m+E\left[\frac{n}{2}\right]+1$. Take

$$y=\begin{cases}0, & i=0, 1, \cdots, m+E\left[\frac{n}{2}\right],\\ 1, & i=m+E\left[\frac{n}{2}\right]+1, \cdots, m+n.\end{cases} \tag{3.1}$$

Concerning this set of y_i, clearly for $R=0$, $m+E\left[\frac{n}{2}\right]+1$ of relations (1.1) are valid. We may further prove that for any $R\in\boldsymbol{R}(m, n)$, no more than $m+E\left[\frac{n}{2}\right]+1$ equalities of (1.1) can hold. If not, there must have $\tilde{R}\in\boldsymbol{R}(m, n)$ such that at least $m+E\left[\frac{n}{2}\right]+2$ equalities of (1.1) are satisfied. Since in (3.1) only $n-E\left[\frac{n}{2}\right]$ of the y_i's are equal to one, there are at most $n-E\left[\frac{n}{2}\right]$ of all y_i which satisfies the relations $\tilde{R}(x_i)=y_i=1$. Consequently, at least $m+E\left[\frac{n}{2}\right]+2-\left(n-E\left[\frac{n}{2}\right]\right)(\geqslant m+1)$ of them have to be zeros. That is, $\tilde{R}$ has at least $m+1$ zeros. So $\tilde{R}=0$. However, we know, if $\tilde{R}=0$, only $m+E\left[\frac{n}{2}\right]+1$ relations in (1.1) are satisfied. This is a contradiction.

(b) We proceed to prove that for any set of $y_i(i=0, 1, \cdots, m+n)$, there exists a rational function $R\in\boldsymbol{R}(m, n)$ such that at least $m+E\left[\frac{n}{2}\right]+1$ relations in (1.1) hold. We need to consider different cases according to the number k of zeros among all y_i.

(i) $k\geqslant m+E\left[\frac{n}{2}\right]+1$. It is sufficient to choose $R=0$.

(ii) $m<k\leqslant m+E\left[\frac{n}{2}\right]$. We may assume

$$y_0=\cdots=y_{k-1}=0, \quad y_i\neq 0, \quad i=k, \cdots, m+n.$$

Let

$$N(x)=\prod_{i=0}^{m-1}(x-x_i).$$

By Lemma 3.1, there exists a polynomial $D(x)$ of degree at most $m+n+1-k$ such that

$$D(x_i)=N(x_i)y_i^{-1},\ i=k, \cdots, m+n \text{ and } D(x_i)\neq 0,\ i=0, 1, \cdots, k-1.$$

Hence, at the points $x_0, x_1, \cdots, x_{m-1}, x_k, \cdots, x_{m+n}$, $R=N/D\in\boldsymbol{R}(m, n)$ satisfies (1.1), i.e., $m+(m+n+1-k)\left(\geqslant m+E\left[\frac{n}{2}\right]+1\right)$ relations in (1.1) are satisfied.

(iii) $0<k<m$. We can presently use the results obtained in § 2 together with the lemmas in this section. If $n^*\leqslant n-E\left[\frac{n}{2}\right]$, Lemma 2.4 implies that for any $R\in \boldsymbol{R}_0(m, n)$, at least $m+n+1-\left(n-E\left[\frac{n}{2}\right]\right)\left(=m+E\left[\frac{n}{2}\right]+1\right)$ relations in (1.1) are satisfied. Now, let $n^*>n-E\left[\frac{n}{2}\right]$ (this implies $n\geqslant 2$). From (2.8), we have

$$r\{C(m, n)\}\geqslant m+n^*+1\geqslant m+n-E\left[\frac{n}{2}\right]+2=(m+n+1)-\left(E\left[\frac{n}{2}\right]-1\right).$$

Therefore, from Lemma 3.2, we can change at most $E\left[\frac{n}{2}\right]-1$ points in $\{(x_i, y_i): 0\leqslant i\leqslant m+n\}$ such that the resultant matrix $C(m, n)$ has rank $m+n+1$. And then, using Lemma 3.3, we can change at most one point of (x_i, y_i) $(i=0, 1, \cdots, m+n)$ so that all the matrices $C_i(m-1, n-1)$ are nonsingular for $i=0, 1, \cdots, m+n$. Let the original points (x_i, y_i), $i=0, 1, \cdots, m+n$, after some of them are changed as described above, be denoted by (x_i', y_i'). Since the number of the changed points is not more than $E\left[\frac{n}{2}\right]$, it is evident that at least $m+n+1-E\left[\frac{n}{2}\right]\left(\geqslant m+E\left[\frac{n}{2}\right]+1\right)$ points in $\{(x_i', y_i'): 0\leqslant i\leqslant m+n\}$ are the same as the original points (x_i, y_i) $(i=0, 1, \cdots, m+n)$. For the points (x_i', y_i'), from the corollary of Theorem 2.4, it follows that there exists a rational function $R\in \boldsymbol{R}(m, n)$ such that

$$R(x_i')=y_i', \qquad i=0, 1, \cdots, m+n$$

hold. Therefore, for this R and the original points (x_i, y_i) $(i=0, 1, \cdots, m+n)$, at least $m+E\left[\frac{n}{2}\right]+1$ relations in (1.1) are satisfied. This completes the proof of the theorem.

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