# ON GLOBAL CONVERGENCE AND APPROXIMATE ITERATION OF THE LINEAR APPROXIMATION METHOD FOR SOLVING VARIATIONAL INEQUALITIES*

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## Abstract

This paper is concerned with the linear approximation method (i.e. the iterative method in which a sequence of vectors is generated by solving certain linearized subproblems) for solving the variational inequality. The global convergent iterative process is proposed by applying the continuation method, and the related problems are discussed. A convergent result is obtained for the approximation iteration (i.e. the iterative method in which a sequence of vectors is generated by solving certain linearized subproblems approximately).

## §1. Introduction

Given a subset $C$ of $R^n$ and a mapping $f$ from $C$ into $R^n$, the variational inequality problem VI$(C, f)$ is to find a vector $x^*\in C$ such that

$$\langle y-x^*, f(x^*)\rangle \geqslant 0, \quad \forall y\in C. \tag{1}$$

An efficient numerical method for solving VI$(C, f)$ is the following iterative scheme.

*Algorithm* 1. Given $y^k\in C$,

$$y^{k+1}=(1-\alpha_k)y^k+\alpha_k x^{k+1}, \quad \alpha_k\in(0, 1], \tag{2}$$

$$x^{k+1} \text{ solves VI}(C, f^k), \tag{3}$$

where

$$f^k(x)=f(y^k)+A(y^k)(x-y^k)$$

and $A(y^k)$ is an $n$ by $n$ matrix.

We regard Algorithm 1 as a linear approximation method. Included in the family of linear approximation methods are the Newton method, the quasi-Newton method, the SOR method, the linearized Jacobi method and the projection method, etc.

For $\alpha_k\equiv1$, Rockafellar has established in [3] a convergence theory for Algorithm 1 by the norm-contraction approach, the vector-contraction approach and the monotone approach. His main result by the norm-contraction approach is the following theorem.

**Theorem 1.** *Assume that*

1) $C\subset R^n$ *is a nonempty closed convex subset;*

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*2) $f: C\to R^n$ and $A: R^n\to R^{n\times n}$ are continuous;*

*3) $x^*$ solves problem $\mathrm{VI}(C, f)$;*

*4) there exists a positive semi-definite metrix $G$ such that $A(x^*)-G$ is positive semi-definite;*

*5) there exists a neighborhood $N$ of $x^*$ such that*

$$\|\widetilde{G}^{-1}[f(x)-f(y)-A(y)(x-y)]\|_{\widetilde{G}}\leqslant b\|x-y\|_{\widetilde{G}}\quad \text{for all } x\in R^n,$$

*where $b<1$, $\widetilde{G}=\frac{1}{2}(G+G^T)$ and $\|\cdot\|_{\widetilde{G}}$ is defined as*

$$\|x\|_{\widetilde{G}}=(x^T\widetilde{G}x)^{\frac{1}{2}}\quad \text{for all } x\in R^n.$$

*Then provided that the initial vector $y^0$ is chosen in a suitable neighborhood of $x^*$, the sequence $\{y^k\}$ generated by Algorithm 1 with $\alpha_k\equiv1$ is well defined and converges to the solution $x^*$. Moreover, there is an $r\in(0, 1)$ such that*

$$\|y^{k+1}-x^*\|_{\widetilde{G}}\leqslant r\cdot\|y^k-x^*\|_{\widetilde{G}}\quad \text{for } k\geqslant0. \tag{4}$$

By using the results of Theorem 1, especially (4), one can prove the following corollary without any difficulty.

**Corollary 1.** Assume that hypotheses 1, 2, 3, 4, 5 in Theorem 1 hold, and assume that

6) $1\geqslant\alpha_k\geqslant\alpha>0,\quad \forall k.$

Then provided that the initial vector $y^0$ is chosen in a suitable neighborhood of $x^*$, the sequence $\{y^k\}$ generated by Algorithm 1 is well defined and converges to the solution $x^*$. Moreover, there is an $r\in(0, 1)$ such that

$$\|y^{k+1}-x^*\|_{\widetilde{G}}\leqslant r\|y^k-x^*\|_{\widetilde{G}}\quad \text{for } k\geqslant0.$$

This paper will deal with two problems about Algorithm 1.

1) Generally, Algorithm 1 is locally convergent. Is there some method to extend it to a global convergent algorithm, or, alternatively, is there some procedure to obtain a sufficiently close starting point $y^0$?

2) When the calculation of (3) is non-exact (i.e. $x^{k+1}$ is an approximation to the solution, not the solution, of $\mathrm{VI}(C, f^k)$, does Algorithm 1 converge and on what condition does it converge?

A question similar to problem 2) was proposed by Rockafellar[4] to the Penalty-Duality method, which is devised for solving variational inequalities, but he pointed out that the answer had been not obtained.

For the first question, applying the continuation method[2] to Algorithm 1, we obtained a global convergent algorithm. The convergence is proved and the related problems are discussed. By adding a very mild (but essential) condition, we obtain a convergent result for the second question.

In the following two sections, we shall deal with the two problems respectively.

## § 2. Global Convergent Iteration

Let a homotopy $H(\cdot, \cdot): C\times[0, 1]\subset R^n\times R^1\to R^n$. We want to solve problem $\mathrm{VI}(C, H(x, 1))$, and the solution of $\mathrm{VI}(C, H(x, 0))$ is given. Assume $x(t)$ continuously depends on $t$.

Let $N$ be a natural number and $m_0, \cdots, m_{N-1}$ be integers. We partition the interval [0, 1] by

$$0=t_0<t_1<\cdots<t_{N-1}<t_N=1. \tag{5}$$

*Algorithm* 2. Let $x^{0,m_0}=x(0)$. Given $x^{i-1,m_{i-1}}$, take $x^{i,0}=x^{i-1,m_{i-1}}$. Compute $x^{i,m_i}$ by (*), $i=1, 2, \cdots, N$.

$$(*)\quad \begin{cases} x^{i,k+1}=(1-\alpha_{i,k})x^{i,k}+\alpha_{i,k}y^{i,k+1}, \\ y^{i,k+1} \text{ solves VI}(C, H^{i,k}(x)), \\ H^{i,k}(x)=H(x^{i,k}, t_i)+A(x^{i,k}, t_i)(x-x^{i,k}) \\ A(x^{i,k}, t_i) \text{ is an } n \text{ by } n \text{ matrix,} \end{cases}$$

where $m_N$ may be taken as a sufficiently large natural number if necessary.

**Theorem 2.** *Suppose that*

1) $C\subset R^n$ *is a nonempty closed convex subset;*

2) $H(.,.): C\times[0, 1]\to R^n$ *and* $A: R^n\times[0, 1]\to R^n\times R^n$ *are continuous;*

3) $x(t)$ *solves* VI$(C, H(\cdot, t))$ $(0\leqslant t\leqslant 1)$, *and* $x(t)$ *continuously depends on* $t$;

4) $G$: $[0, 1]\subset R^1\to R^n\times R^n$ *is continuous, and* $G(t)$ *is a positive definite matrix with* $A(x(t), t)-G(t)$ *being positive semi-definite for any* $t\in[0, 1]$;

5) *there are two positive numbers* $b$ *and* $\delta$ *with* $b<1$ *such that for each* $t\in[0, 1]$ *we have*

$$\|\tilde{G}(t)^{-1}[H(x, t)-H(y, t)-A(y, t)(x-y)]\|_{\tilde{G}(t)}\leqslant b\|x-y\|_{\tilde{G}(t)};$$
$$x, y\in S(x(t), \delta)\cap C;$$

6) $0<\alpha\leqslant\alpha_{i,k}\leqslant 1$.

*Then there exists a partition* (5) *of* [0, 1] *and integers* $m_0, m_1, \cdots, m_{N-1}$ *such that the entire sequence* $\{x^{i,k}\}$ *generated by Algorithm* 2 *is well defined and that*

$$\lim_{k\to\infty} x^{N,k}=x(1).$$

*Proof.* For any $t\in[0, 1]$, it follows from corollary 1 that there exists a neighborhood $S(x(t), \delta_t)$ of $x(t)$ such that for each point $x^{t,0}\in S(x(t), \delta_t)\cap C$, the sequence $\{x^{t,k}\}$ generated by the following iteration

$$(**)\quad \begin{cases} x^{t,k+1}=(1-\alpha_{t,k})x^{t,k}+\alpha_{t,k}y^{t,k+1}, \quad \alpha_{t,k}\in[\alpha, 1], \\ y^{t,k+1} \text{ solves VI}(C, H^{t,k}(x)), \\ H^{t,k}(x)=H(x^{t,k}, t)+A(x^{t,k}, t)(x-x^{t,k}) \end{cases}$$

converges to $x(t)$.

Denote inf $\{\delta_t: t\in[0, 1]\}$ by $\bar{\delta}$. From condition 5) and the procedure of $\delta_t$'s generation, which was described in the proof of Theorem 1 in [3], we know $\bar{\delta}>0$.

Now, let a partition (5) be chosen such that

$$\max_{0\leqslant i\leqslant N-1}\|x(t_{i+1})-x(t_i)\|\leqslant\bar{\bar{\delta}}<\bar{\delta}.$$

Because $x(0)=x^{1,0}$ and

$$\|x(0)-x(t_1)\|\leqslant\bar{\bar{\delta}}<\bar{\delta}\leqslant\delta_{t_1}$$

$\{x^{1,k}\}$ converges to $x(t_1)$, which is generated by (**) with $t=t_1$ (i.e. Algorithm 2). So there is a sufficiently large number $m_1$ such that

$$\|x^{1,m_1}-x(t_1)\|\leqslant\bar{\delta}-\bar{\bar{\delta}}.$$

As a result, we obtain the second initial vector $x^{2,0}=x^{1,m_1}$–which satisfies

$$\|x^{2,0}-x(t_2)\|=\|x^{1,m_1}-x(t_1)+x(t_1)-x(t_2)\|\leqslant\delta-\bar{\bar{\delta}}+\bar{\bar{\delta}}=\delta\leqslant\delta_{t_2}.$$

Consequently, $\{x^{2,k}\}$ converges to $x(t_2)$, $\cdots$.

We continue the process until $x^{N,0}$ is obtained. Proceeding in the same way one can show

$$x^{N,k}\to x(1), \quad k\to+\infty.$$

Q.E.D.

The general convergence of Algorithm 2 is proved, but there is a problem: How to guarantee the continuity of the solution $x(t)$ of $\mathrm{VI}(C, H(x, t))$? The following two theorems will treat this problem for Newton homotopy and convex combined homotopy respectively.

**Theorem 3.** *Assume that*

1) *$C\subset R^n$ is a nonempty closed convex subset;*

2) *$f$: $C\to R^n$ is continuous and strongly monotone, i.e. there exists a positive number $r$ such that*

$$\langle f(x)-f(y), x-y\rangle\geqslant r\|x-y\|^2, \quad x, y\in C;$$

3) *let $x^0\in C$ and define*

$$H(x, t)=f(x)+(t-1)f(x^0), \quad t\in[0, 1]. \tag{6}$$

*Then there is a unique continuous curve $x(t)$ $(0\leqslant t\leqslant 1)$ in $C$ such that for each $t\in[0, 1]$ we have*

$$\langle x-x(t), H(x(t), t)\rangle\geqslant 0, \quad \forall x\in C.$$

*Proof.* For each $t\in[0, 1]$, it is easy to check that $H(x, t)$ is strongly monotone about $x$. By a classical result of Stampacchia[1] problem $\mathrm{VI}(C, H(x, t))$ has a unique solution $x(t)$ $(0\leqslant t\leqslant 1)$. In what follows, we will prove that $x(t)$ continuously depends on $t$.

For any $t$ and $\alpha$ in $[0, 1]$, we have

$$\langle x(\alpha)-x(t), H(x(t), t)\rangle\geqslant 0, \tag{7}$$

$$\langle x(t)-x(\alpha), H(x(\alpha), \alpha)\rangle\geqslant 0. \tag{8}$$

Since

$$H(x(t), t)-H(x(\alpha), \alpha)=f(x(t))-f(x(\alpha))+(t-\alpha)f(x^0)$$

(7) + (8) gives

$$\langle x(\alpha)-x(t), f(x(t))-f(x(\alpha))+(t-\alpha)f(x^0)\rangle\geqslant 0.$$

Hence

$$\langle x(\alpha)-x(t), (t-\alpha)f(x^0)\rangle\geqslant\langle x(\alpha)-x(t), f(x(\alpha))-f(x(t))\rangle \geqslant r\|x(\alpha)-x(t)\|^2.$$

Therefore

$$\|x(\alpha)-x(t)\|\leqslant\|f(x^0)\|\cdot\frac{1}{r}|\alpha-t|.$$

As a result, $x(t)$ is Lipschitz continuous in $t$. Q.E.D.

**Theorem 4.** *Assume that*

1) *$C\subset R^n$ is a nonempty closed convex subset:*

2) $f$: $C\to R^n$ *is continuous and monotone;*

3) $x^*$ *solves* VI$(C, f)$ *and there exists an* $r>0$ *such that*

$$\langle x-x^*, f(x)-f(x^*)\rangle \geqslant r\|x-x^*\|^2, \quad \forall x\in C;$$

4) *Let* $x^0\in C$ *and define*

$$H(x, t)=tf(x)+(1-t)(x-x^0). \tag{9}$$

*Then there is a unique continuous curve* $x(t)$ $(0\leqslant t\leqslant 1)$ *in* $C$ *such that, for each* $t\in[0, 1]$, *we have*

$$\langle x-x(t), H(x(t), t)\rangle \geqslant 0, \quad \forall x\in C.$$

***Proof.*** First, we prove VI$(C, H(x, t))$ has a unique solution for each $t\in[0, 1]$. When $t=1$, condition 3, quarantees that VI$(C, H(x, 1))$ has a unique solution $x(1)=x^*$. When $t\in[0, 1]$, we have

$$\begin{aligned}\langle x-y, H(x, t)-H(y, t)\rangle &= \langle x-y, t(f(x)-f(y))+r(t)(x-y)\rangle\\ &\geqslant r(t)\cdot\|x-y\|^2, \quad \forall x, y\in C,\end{aligned}$$

where $r(t)=(1-t)$. By a result of [1] problem VI$(C, H(x, t))$ has a unique solution $x(t)$. In the sequel, we shall prove the continuity of $x(t)$ $(0\leqslant t\leqslant 1)$. For $t$, $\alpha\in[0, 1]$, we have

$$\langle x(\alpha)-x(t), H(x(t), t)\rangle \geqslant 0, \tag{10}$$

$$\langle x(t)-x(\alpha), H(x(\alpha), \alpha)\rangle \geqslant 0. \tag{11}$$

Then (10)+(11) yields

$$\begin{aligned}&\langle x(\alpha)-x(t), (t-\alpha)f(x(t))+\alpha(f(x(t))-f(x(\alpha))\\ &\quad +(r(t)-r(\alpha))x(t)+r(\alpha)(x(t)-x(\alpha))+(r(\alpha)-r(t))x^0\rangle \geqslant 0.\end{aligned}$$

Hence

$$\begin{aligned}&\langle x(\alpha)-x(t), (t-\alpha)f(x(t))+(r(t)-r(\alpha))x(t)+(r(\alpha)-r(t))x^0\rangle\\ &\quad \geqslant \langle x(\alpha)-x(t), \alpha(f(x(\alpha))-f(x(t)))+r(\alpha)(x(\alpha)-x(t))\rangle\\ &\quad \geqslant [\alpha\tilde{r}+r(\alpha)]\cdot\|x(\alpha)-x(t)\|^2,\end{aligned}$$

where

$$\tilde{r}=\begin{cases} r, & t=1,\\ 0, & t<1.\end{cases}$$

Therefore

$$\begin{aligned}&\{|t-\alpha|\cdot\|f(x(t))\|+|r(\alpha)-r(t)|\cdot(\|x(t)\|+\|x^*\|)\}\cdot\|x(\alpha)-x(t)\|\\ &\quad \geqslant [\alpha\tilde{r}+r(\alpha)]\cdot\|x(\alpha)-x(t)\|^2.\end{aligned}$$

Noticing $r(\alpha)-r(t)=t-\alpha$, we obtain

$$(\alpha\tilde{r}+r(\alpha))\cdot\|x(\alpha)-x(t)\| \leqslant \{\|f(x(t))\|+\|x(t)\|+\|x^0\|\}\cdot|t-\alpha|. \tag{12}$$

Fixing $t\in[0, 1]$, we let $\alpha$ tend to $t$ in (12) and obtain

$$(t\tilde{r}+r(t))\cdot\overline{\lim_{\alpha\to t}}\|x(\alpha)-x(t)\|=0.$$

Since

$$t\tilde{r}+r(t)=\begin{cases} r, & t=1,\\ 1-t, & t<1\end{cases}$$

we have

$$\lim_{\alpha\to t}\|x(\alpha)-x(t)\|=0.$$

Consequently, $x(t)$ is continuous on $[0, 1]$. Q.E.D.

Combining Theorem 2 with Theorem 3 (or Theorem 4), we can obtain the following theorem.

**Theorem 5.** *Assume that*

1) *$C$ is a nonempty closed convex subset;*

2) *$f: C_1\to R^n$ is continuously differentiable and strongly monotone, where $C_1\subset R^n$ is an open convex subset containing $C$;*

3) *Let $x^0\in C$ and let $H(x, t)$ be defined by (6) or (9).*

*Then* 1) *there is a unique continuous curve $x(t)$ $(0\leqslant t\leqslant 1)$ in $C$ such that for each $t\in[0, 1]$, we have*

$$\langle x-x(t), H(x(t), t)\rangle\geqslant 0 \quad \forall x\in C;$$

2) *if we take $A(x, t)=\partial_x H(x, t)$ in Algorithm 2, there exists a partition (5) of $[0, 1]$ and integers $m_0, m_1, \cdots, m_{N-1}$ such that the entire sequence $\{x^{i,k}\}$ generated by Algorithm 2 with $1\geqslant\alpha_{i,k}\geqslant\alpha>0$ is well defined and that*

$$\lim_{k\to+\infty} x^{N,k}=x(1).$$

*Proof.* The first conclusion follows from Theorem 3 or Theorem 4. To prove the second conclusion, it is sufficient to check that all the conditions of Theorem 2 hold.

Because $f$ is continuously differentiable and strongly monotone on $C_1$ by a result of [2], there exists $\bar\beta>0$ such that

$$y^T\nabla f(x)y\geqslant\bar\beta y^Ty, \quad \forall x\in C, y\in R^n.$$

Taking $\beta=\min\{\bar\beta, 1\}$, we have

$$y^T\partial_x H(x, t)y\geqslant\beta y^Ty, \quad \forall x\in C, y\in R^n.$$

Choosing $G(t)\equiv\beta I$, one may check that $A(x(t), t)-G(t)=H(x(t), t)-\beta I$ is positive semi-definite.

For the continuity of $x(t)$, $D_0=\{x(t): 0\leqslant t\leqslant 1\}\subset C$ is a compact subset. There exists a convex compact subset $D$ such that

$$S(D_0, 1)\cap C\subset D\subset C.$$

Because $\partial_x H(x, t)$ is continuous on $D\times[0, 1]$, for $0<\varepsilon<\beta$ there exists $0<\delta<1$ such that

$$\|x-y\|\leqslant\delta,\ x, y\in D\Rightarrow\|\partial_x H(x, t)-\partial_x H(y, t)\|\leqslant\varepsilon, \quad t\in[0, 1].$$

Therefore

$$\begin{aligned}&\|H(x, t)-H(y, t)-\partial_x H(y, t)(x-y)\|\\ &\leqslant\sup_{0<\sigma<1}\|\partial_x H(y+\sigma(x-y), t)-\partial_x H(x, t)\|\cdot\|x-y\|\\ &\leqslant\varepsilon\cdot\|x-y\|, \quad \forall y\in D, x\in S(y, \delta)\cap D.\end{aligned}$$

Hence

$$\begin{aligned}&\|\tilde G(t)^{-1}[H(x, t)-H(y, t)-\partial_x H(x, t)(x-y)]\|_{\tilde G(t)}\\ &\leqslant\varepsilon\beta^{-1}\|x-y\|_{\tilde G(t)}, \quad \forall x, y\in S(x(t), \delta/2), t\in[0, 1].\end{aligned}$$

For the choice of $\varepsilon$, we have $\varepsilon\beta^{-1}<1$. As a result, all the conditions of Theorem 2 hold, and the proof is completed. Q.E.D.

## § 3. Approximate Iteration

*Algorithm* 3. Given $v^k \in C$,

$$v^{k+1} = (1-\alpha_k) v^k + \alpha_k u^{k+1}, \quad \alpha_k \in (0, 1],$$

$u^{k+1}$ is an approximation to the solution, of $\mathrm{VI}(C, f^k)$, where $f^k(x) = f(v^k) + A(v^k)(x - v^k)$ and $A(v^k)$ is an $n$ by $n$ matrix.

When $u^{k+1}$ solves $\mathrm{VI}(C, f^k)$, it is nothing but Algorithm 1. Denote $A(v^k)$ by $A_k$ and $\frac{1}{2}(A_k + A_k^T)$ by $\widetilde{A}_k$. As assumed in Corollary 1, we suppose $A_k$ is positive definite. Consequently, $\mathrm{VI}(C, f^k)$ has a unique solution $x^{k+1}$. Let

$$\delta^k = \|u^{k+1} - x^{k+1}\|.$$

**Theorem 6.** Assume that

1) $C \subset R^n$ is a nonempty closed convex subset;

2) $f: C \to R^n$ is continuous;

3) $x^*$ solves problem $\mathrm{VI}(C, f)$;

4) in Algorithm 3, for each $k$ suppose

a) $m_k^2 y^T y \leqslant y^T A_k y \leqslant M_k^2 y^T y, \quad y \in R^n$,

where $M_k \geqslant m_k \geqslant m > 0$;

b) $\|\widetilde{A}_k^{-1}[f(x^*) - f(v^k) - A_k(x^* - v^k)]\|_k \leqslant \lambda_k \|x^* - v^k\|_k$,

*where* $\lambda_k \geqslant 0$ *and* $\|\cdot\|$ *is defined as*

$$\|x\| = (x^T \widetilde{A}_k x)^{\frac{1}{2}}, \quad \forall x \in R^n;$$

c) $1 \geqslant \alpha_k \geqslant \alpha > 0$, $\lambda_k \cdot \dfrac{M_k}{m_k} \leqslant \lambda < 1$;

5) $\delta^k = \|u^{k+1} - x^{k+1}\| \to 0$ *as* $k \to +\infty$.

*Then*

$$v^k \to x^*, \quad as\ k \to +\infty.$$

The following lemma is useful for the proof of this theorem.

**Lemma.** *For each fixed* $k$, *there exist* $\varepsilon_i^k \geqslant 0$, $i = 1, 2$, *such that*

$$\langle x - u^{k+1}, f^k(u^{k+1})\rangle + \varepsilon_1^k \cdot \|x - u^{k+1}\| + \varepsilon_2^k \geqslant 0, \quad \forall x \in C \tag{13}$$

*and*

$$\varepsilon_1^k + \varepsilon_2^k \to 0 \quad as\ \delta^k \to 0,\ k\ is\ a\ fixed\ number. \tag{14}$$

*Proof.* For any $x \in C$

$$\langle x - u^{k+1}, f^k(u^{k+1})\rangle$$
$$= \langle x - x^{k+1}, f^k(x^{k+1})\rangle + \langle x^{k+1} - u^{k+1}, f^k(u^{k+1})\rangle + \langle x - u^{k+1}, f^k(x^{k+1}) - f^k(u^{k+1})\rangle.$$

Because $x^{k+1}$ solves $\mathrm{VI}(C, f^k)$, we have

$$\langle x - u^{k+1}, f^k(u^{k+1})\rangle \geqslant \langle x^{k+1} - u^{k+1}, f^k(u^{k+1})\rangle + \langle x - u^{k+1}, f^k(x^{k+1}) - f^k(u^{k+1})\rangle$$
$$\geqslant -\|f^k(u^{k+1})\| \cdot \|x^{k+1} - u^{k+1}\| - \|x - u^{k+1}\| \cdot \|f^k(x^{k+1}) - f^k(u^{k+1})\|.$$

If we take

$$\varepsilon_1^k = \|f^k(x^{k+1}) - f^k(u^{k+1})\|, \quad \varepsilon_2^k = \|f^k(u^{k+1})\| \cdot \|x^{k+1} - u^{k+1}\|$$

(13) is obtained. Owing to the continuity of $f^k$, (14) is true.

*Proof of Theorem* 6. For $x^*$ solving VI$(C, f)$, we have

$$\langle u^{k+1}-x^*, f(x^*)\rangle \geqslant 0. \tag{15}$$

Taking $x=x^*$ in (13), we obtain

$$\langle x^*-u^{k+1}, f^k(u^{k+1})\rangle+\varepsilon_1^k\|x^*-u^{k+1}\|+\varepsilon_2^k\geqslant 0. \tag{16}$$

(15)+(16) gives

$$\langle u^{k+1}-x^*, f(x^*)-f(v^k)-A_k(x^*-v^k)+A_k(x^*-u^{k+1})\rangle+\varepsilon_1^k\cdot\|x^*-u^{k+1}\|+\varepsilon_2^k\geqslant 0.$$

Therefore

$$\begin{aligned}&\|u^{k+1}-x^*\|_k\cdot\|\widetilde{A}_k^{-1}[f(x^*)-f(v^k)-A_k(x^*-v^k)]\|_k+\varepsilon_1^k\|x^*-u^{k+1}\|+\varepsilon_2^k\\ &\geqslant\langle u^{k+1}-x^*, A_k(u^{k+1}-x^*)\rangle.\end{aligned}$$

Noticing condition 4) and

$$z^TA_kz=z^T\widetilde{A}_kz, \quad \forall z\in R^n$$

we obtain

$$\|u^{k+1}-x^*\|_k\cdot\lambda_k\|v^k-x^*\|_k+\varepsilon_1^k\|x^*-u^{k+1}\|+\varepsilon_2^k\geqslant\|u^{k+1}-x^*\|_k^2.$$

Hence

$$\begin{aligned}\|u^{k+1}-x^*\|_k^2&\leqslant\lambda_k\|v^k-x^*\|_k\cdot\|u^{k+1}-x^*\|_k^2+\varepsilon_1^k\cdot\|\widetilde{A}_k^{-\frac{1}{2}}[x^*-u^{k+1}]\|_k+\varepsilon_2^k\\ &\leqslant[\lambda_k\|v^k-x^*\|_k+\varepsilon_1^k\|\widetilde{A}_k^{-\frac{1}{2}}\|_k]\cdot\|x^*-u^{k+1}\|_k+\varepsilon_2^k\\ &=[\lambda_k\|v^k-x^*\|_k+\varepsilon_1^k\|\widetilde{A}_k^{-\frac{1}{2}}\|]\cdot\|x^*-u^{k+1}\|_k+\varepsilon_2^k.\end{aligned}$$

Thus

$$\|u^{k+1}-x^*\|_k\leqslant\lambda_k\|v^k-x^*\|_k+\varepsilon_1^k\|A_k^{-\frac{1}{2}}\|+(\varepsilon_2^k)^{\frac{1}{2}}.$$

Noticing the definition of $\|\cdot\|_k$ we have

$$\begin{aligned}\|u^{k+1}-x^*\|&\leqslant\lambda_k\cdot\frac{M_k}{m_k}\|v^k-x^*\|+\frac{1}{m_k}(\varepsilon_1^k\cdot\|A_k^{-\frac{1}{2}}\|+(\varepsilon_2^k)^{\frac{1}{2}})\\ &\leqslant\lambda\|v^k-x^*\|+\frac{1}{m}\Big(\varepsilon_1^k\cdot\frac{1}{m}+(\varepsilon_2^k)^{\frac{1}{2}}\Big).\end{aligned}$$

Let

$$\varepsilon^k=\frac{1}{m^2}\varepsilon_1^k+\frac{1}{m}(\varepsilon_2^k)^{\frac{1}{2}}.$$

Then

$$\begin{aligned}\|v^{k+1}-x^*\|&\leqslant(1-\alpha_k)\|v^k-x^*\|+\alpha_k\|u^{k+1}-x^*\|\leqslant[1-\alpha_k+\alpha_k\lambda]\cdot\|v^k-x^*\|+\alpha_k\varepsilon^k\\ &\leqslant[1-\alpha+\lambda\alpha]\cdot\|v^k-x^*\|+\varepsilon^k\triangleq r\|v^k-x^*\|+\varepsilon^{k-1}+\varepsilon^k\\ &\leqslant r^2\|v^{k-1}-x^*\|+r\varepsilon^{k-1}+\varepsilon^k\leqslant\cdots\leqslant r^{k+1}\|v^0-x^*\|+\sum_{i=0}^{k}r^i\cdot\varepsilon^{k-i}.\end{aligned}$$

Noticing

$$0<r<1$$

and

$$\varepsilon^k\to 0 \quad \text{as } k\to+\infty$$

applying the Toeplitz lemma, we obtain

$$v^k\to x^*, \quad k\to+\infty.$$

Q.E.D.

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