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# THE UNSOLVABILITY OF MULTIPLICATIVE INVERSE EIGENVALUE PROBLEMS ALMOST EVERYWHERE*

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## Abstract

The idea and technique used in [7] are applied to the multiplicative inverse eigenvalue problems as well. Some sufficient and necessary conditions that the multiplicative inverse eigenvalue problems be unsolvable almost everywhere are given. The results are similar to those of [7], but the proofs are more complicated.

## § 1. Introduction

The multiplicative inverse eigenvalue problems for real matrices are the following (see [2], [4]):

**Problem M-1.** Given an $n\times n$ positive definite symmetric matrix $A$, $k$ non-zero real numbers $\lambda_1, \cdots, \lambda_k$ and $k+1$ nonnegative integers $r_0, r_1, \cdots, r_k$ satisfying $r_0+r_1+\cdots+r_k=n$ $(k\geqslant 1)$, find a real $n\times n$ diagonal matrix $C=\mathrm{diag}(c_1, \cdots, c_n)$ such that the matrix $CA$ has a zero eigenvalue of multiplicity $r_0$ and eigenvalues $\lambda_1, \cdots, \lambda_k$ of multiplicity $r_1, \cdots, r_k$, respectively.

**Problem GM-1.** Given $m$ real $n\times n$ symmetric matrices $A_1, \cdots, A_m$, $k$ non-zero real numbers $\lambda_1, \cdots, \lambda_k$ and $k+1$ nonnegative integers $r_0, r_1, \cdots, r_k$ satisfying $r_0+r_1+\cdots+r_k=n$ $(k\geqslant 1)$, find $m$ real numbers $c_1, \cdots, c_m$ such that the matrix $c_1A_1+\cdots+c_mA_m$ has a zero eigenvalue of multiplicity $r_0$ and eigenvalues $\lambda_1, \cdots, \lambda_k$ of multiplicity $r_1, \cdots, r_k$, respectively.

**Problem M-2.** Given a real $n\times n$ nonsingular matrix $A$, $k$ non-zero real numbers $\lambda_1, \cdots, \lambda_k$ and $k+1$ nonnegative integers $r_0, r_1, \cdots, r_k$ satisfying $r_0+r_1+\cdots+r_k=n(k\geqslant 1)$, find a real $n\times n$ diagonal matrix $C=\mathrm{diag}(c_1, \cdots, c_n)$ such that the matrix $CA$ is diagonalizable and has a zero eigenvalue of multiplicity $r_0$ and eigenvalues $\lambda_1, \cdots, \lambda_k$ of multiplicity $r_1, \cdots, r_k$, respectively.

**Problem GM-2.** Given $m$ real $n\times n$ matrices $A_1, \cdots, A_m$, $k$ non-zero real numbers $\lambda_1, \cdots, \lambda_k$ and $k+1$ nonnegative integers $r_0, r_1, \cdots, r_k$ satisfying $r_0+r_1+\cdots+r_k=n$ $(k\geqslant 1)$, find $m$ real numbers $c_1, \cdots, c_m$ such that the matrix $c_1A_1+\cdots+c_mA_m$ is diagonalizable and has a zero eigenvalue of multiplicity $r_0$ and eigenvalues $\lambda_1, \cdots, \lambda_k$ of multiplicity $r_1, \cdots, r_k$, respectively.

Problem M-1 is a classical multiplicative inverse eigenvalue problem (see [4]). Problems GM-1 and GM-2 are general multiplicative inverse eigenvalue problems for real matrices (see [2]). These problems have been studied by several authors,

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This paper is a continuation of [7]. In this paper we give some sufficient and necessary conditions that the problems M-1, GM-1, M-2 and GM-2 be unsolvable almost everywhere (a.e.), respectively.

Notation. The symbol $\mathbb{R}^{m\times n}$ denotes the set of real $m\times n$ matrices, $\mathbb{R}^n=\mathbb{R}^{n\times 1}$ and $\mathbb{R}=\mathbb{R}^1$. $I^{(n)}$ is the $n\times n$ identity matrix and $O$ is the null matrix. $\mathscr{R}(A)$ stands for the column space of $A$. The superscript $T$ is for transpose, and

$$S\mathbb{R}^{n\times n}=\{A\in\mathbb{R}^{n\times n}:\ A^T=A\},\ \mathbb{O}^{n\times n}=\{A\in\mathbb{R}^{n\times n}:\ A^TA=I\}$$

and

$$S\mathbb{R}_+^{n\times n}=\{A\in S\mathbb{R}^{n\times n}:\ A \text{ is positive definite}\}.$$

Besides, for $A=(a_{ij})\in\mathbb{R}^{n\times n}$ we write

$$|A|=(|a_{ij}|),\ k_1(A)=\max_{1<j<n}\Big(\sum_{\substack{i=1\\ i\neq j}}^{n}|a_{ij}|\Big)$$

and

$$k_2(A)=\max_{1<j<n}\Big(\sum_{\substack{i=1\\ i\neq j}}^{n}a_{ij}^2\Big)^{1/2}.$$

Now we define the unsolvability of multiplicative inverse eigenvalue problems a.e.

**Definition 1.1.** *Problem M-1 is said to be unsolvable almost everywhere (u.s.a.e.) if the set of matrices $A\in S\mathbb{R}_+^{n\times n}$ and vectors $\lambda=(\lambda_1,\cdots,\lambda_k)^T\in\mathbb{R}^k$ at which it is solvable has measure zero in the open set $S\mathbb{R}_+^{n\times n}\times\mathbb{R}^k$ of the product vector space $S\mathbb{R}^{n\times n}\times\mathbb{R}^k$.*

**Definition 1.2.** *Problem GM-1 is said to be u.s.a.e. if the set of matrices $A_1,\cdots,A_m\in S\mathbb{R}^{n\times n}$ and vectors $\lambda=(\lambda_1,\cdots,\lambda_k)^T\in\mathbb{R}^k$ at which it is solvable has measure zero in the product vectors space $\underbrace{S\mathbb{R}^{n\times n}\times\cdots\times S\mathbb{R}^{n\times n}}_{m}\times\mathbb{R}^k$.*

**Definition 1.3.** *Problem M-2 is said to be u.s.a.e. if the set of matrices $A\in\mathbb{R}^{n\times n}$ and vectors $\lambda=(\lambda_1,\cdots,\lambda_k)^T\in\mathbb{R}^k$ at which it is solvable has measure zero in the product vector space $\mathbb{R}^{n\times n}\times\mathbb{R}^k$.*

**Definition 1.4.** *Problem GM-2 is said to be u.s.a.e. if the set of matrices $A_1,\cdots,A_m\in\mathbb{R}^{n\times n}$ and vectors $\lambda=(\lambda_1,\cdots,\lambda_k)^T\in\mathbb{R}^k$ at which it is solvable has measure zero in the product vector space $\underbrace{\mathbb{R}^{n\times n}\times\cdots\times\mathbb{R}^{n\times n}}_{m}\times\mathbb{R}^k$.*

## § 2. Main Results

**Theorem 2.1.** *Problem M-1 is u.s.a.e. if and only if*

$$\max\{r_1,\cdots,r_k\}\geq 1. \tag{2.1}$$

**Theorem 2.2.** *Problem GM-1 is u.s.a.e. if*

$$n-m+\frac{r(r-1)}{2}\geq 0, \tag{2.2}$$

*where $r=\max\{r_0,r_1,\cdots,r_k\}$. In addition, if $m=n$, then $r>1$ is a sufficient and necessary condition for the unsolvability of Problem GM-1 a.e.*

**Theorem 2.3.** *Problem M-2 is u.s.a.e. if and only if*

$$\max\{r_1, \cdots, r_k\} > 1. \tag{2.3}$$

**Theorem 2.4.** *Problem GM-2 is u.s.a.e. if*

$$n - m + r(r-1) > 0, \tag{2.4}$$

*where $r = \max\{r_0, r_1, \cdots, r_k\}$. In addition, if $m = n$, then $r > 1$ is a sufficient and necessary condition for the unsolvability of Problem GM-2 a.e.*

## § 3. Proofs of Theorem 2.1—Theorem 2.4

The proofs of Theorem 2.1—Theorem 2.4 will be based on the following lemmas (see Theorem 3.1, Theorem 3.2 and Lemma 3.1 of [7], pp. 34—42 of [1] and pp. 45—55 of [6]).

**Lemma 3.1.** *Let $f = (f_1, \cdots, f_r)$ be a differentiable vector-valued function defined in $\mathbb{R}^n$, and let $\mathfrak{M} \subset \mathbb{R}^n$ be the set of points $x = (\xi_1, \cdots, \xi_n)^T \in \mathbb{R}^n$ such that*

$$f_1(x) = 0, \cdots, f_r(x) = 0.$$

*Assume that for each point $x \in \mathfrak{M}$ the matrix*

$$\frac{\partial f}{\partial x} = \begin{pmatrix} \dfrac{\partial f_1}{\partial \xi_1} & \cdots & \dfrac{\partial f_r}{\partial \xi_1} \\ \vdots & & \vdots \\ \dfrac{\partial f_1}{\partial \xi_n} & \cdots & \dfrac{\partial f_r}{\partial \xi_n} \end{pmatrix},$$

*where each partial derivative is evaluated at $x$ has rank $r$. Then $\mathfrak{M}$ is an $n-r$-dimensional submanifold of $\mathbb{R}^n$.*

**Lemma 3.2.** *Let $\mathfrak{M}$ be an $m$-dimensional differentiable submanifold of an $n$-dimensional Euclidean space $\mathscr{E}$, $\mathscr{K}$ be a $k$-dimensional Euclidean space, $k < n$, and let $F$ be a differentiable mapping of $\mathfrak{M} \to \mathscr{K}$ defined by*

$$\eta_1 = \xi_1, \cdots, \eta_k = \xi_k \text{ for } x = (\xi_1, \cdots, \xi_n)^T \in \mathfrak{M} \text{ and } y = (\eta_1, \cdots, \eta_k)^T \in \mathscr{K}.$$

*Then $F(\mathfrak{M})$ is a set of measure zero in $\mathscr{K}$ (represented by meas $F(\mathfrak{M}) = 0$) if $m < k$.*

Now we prove Theorem 2.1—Theorem 2.4.

*Proof of Theorem* 2.1.

1) Suppose that Problem M-1 is solvable at

$$A = (a_{ij}) \in SR_+^{n \times n}, \quad \lambda = (\lambda_1, \cdots, \lambda_k)^T \in \mathbb{R}^k \tag{3.1}$$

with $\lambda_1 > \cdots > \lambda_t > 0 > \lambda_{t+1} > \cdots > \lambda_k$. Then there exist an $n'$-dimensional principal submatrix $A'$ of $A$ and a real $n' \times n'$ non-singular diagonal matrix $C' = \operatorname{diag}(c_1, \cdots, c_{n'})$ such that the matrix $C'A'$ has eigenvalues $\lambda_1, \cdots, \lambda_k$ of multiplicity $r_1, \cdots, r_k$ respectively, in which $n' = r_1 + \cdots + r_k$. Let $n_1 = r_1 + \cdots + r_t$. Observe that by the law of inertia for quadratic forms (see [3], p. 297) the matrix $\operatorname{diag}(I^{(t)}, -I^{(n'-t)}) A'$ has $t$ positive eigenvalues and $n' - t$ negative eigenvalues. Hence there are $n_1$ positive numbers and $n' - n_1$ negative numbers in the set $\{c_1, \cdots, c_{n'}\}$. Without loss of generality we may assume that

$$c_1, \cdots, c_{n_1} > 0, \quad c_{n_1+1}, \cdots, c_{n'} < 0.$$

First we consider the case of

$$A' = \begin{pmatrix} a_{11} & \cdots & a_{1n'} \\ \vdots & & \vdots \\ a_{n'1} & \cdots & a_{n'n'} \end{pmatrix}. \tag{3.2}$$

Let

$$d_1 = c_1^{-\frac{1}{2}}, \cdots, d_{n_1} = c_{n_1}^{-\frac{1}{2}}, \quad d_{n_1+1} = (-c_{n_1+1})^{-\frac{1}{2}}, \cdots, d_{n'} = (-c_{n'})^{-\frac{1}{2}},$$
$$D = \mathrm{diag}(d_1, \cdots, d_{n'}), \quad \Lambda = \mathrm{diag}(\lambda_1 I^{(r_1)}, \cdots, \lambda_k I^{(r_k)}) \tag{3.3}$$

and

$$J = \mathrm{diag}(I^{(n_1)}, -I^{(n'-n_1)}). \tag{3.4}$$

Then there exist nonsingular matrices $Y$, $Z \in \mathbb{R}^{n' \times n'}$ such that

$$A' = DJY\Lambda Z^T D, \tag{3.5}$$

where

$$Y = (Y_1, Y_2, \cdots, Y_k), \ Z = (Z_1, Z_2, \cdots, Z_k), \quad Y_i, Z_i \in \mathbb{R}^{n' \times r_i}, \ i = 1, \cdots, k, \tag{3.6}$$
$$Z^T Y = I^{(n')} \tag{3.7}$$

and

$$Z_i^T Z_i = I^{(r_i)}, \quad i = 1, \cdots, k. \tag{3.8}$$

It follows from (3.5) and $A'^T = A'$ that

$$Z^T JZ\Lambda = \Lambda Z^T JZ, \quad Y^T JY\Lambda = \Lambda Y^T JY;$$

therefrom we have

$$Z_i^T JZ_j = 0, \quad Y_i^T JY_j = 0, \quad 1 \leqslant i < j \leqslant k. \tag{3.9}$$

From (3.7) and (3.9) we see that

$$\mathscr{R}(Y_i) \perp \mathscr{R}((Z_1, \cdots, Z_{i-1}, Z_{i+1}, \cdots, Z_k)), \quad i = 1, \cdots, k$$

and

$$\mathscr{R}(JZ_i) \perp \mathscr{R}((Z_1, \cdots, Z_{i-1}, Z_{i+1}, \cdots, Z_k)), \quad i = 1, \cdots, k,$$

and so there exist nonsingular matrices $R_i \in \mathbb{R}^{r_i \times r_i}$ such that

$$Y_i = JZ_i R_i, \quad i = 1, \cdots, k. \tag{3.10}$$

Combining with (3.7) we get

$$Z_i^T JZ_i R_i = I^{(r_i)}, \quad i = 1, \cdots, k. \tag{3.11}$$

The relations (3.11) show that $R_i \in S\mathbb{R}^{r_i \times r_i}$ for $i = 1, \cdots, k$. We decompose

$$R_i = Q_i \Omega_i Q_i^T, \quad i = 1, \cdots, k, \tag{3.12}$$

where

$$Q_i \in \mathbb{O}^{r_i \times r_i}, \quad \Omega_i = \mathrm{diag}(\omega_{i1}, \cdots, \omega_{i, r_i}), \quad \prod_{j=1}^{r_i} \omega_{ij} \neq 0, \quad i = 1, \cdots, k. \tag{3.13}$$

Let

$$U_i = Z_i Q_i, \quad i = 1, \cdots, k. \tag{3.14}$$

Then from (3.10), (3.12) and (3.14)

$$Y_i Z_i^T = JU_i \Omega_i U_i^T, \quad i = 1, \cdots, k. \tag{3.15}$$

Without loss of generality we may assume that $r_1 = \max\{r_1, \cdots, r_k\}$. It follows from (3.5), (3.7) and (3.15) that

$$A'=DJ[\lambda_1 I+(\lambda_2-\lambda_1)Y_2Z_2^T+\cdots+(\lambda_k-\lambda_1)Y_kZ_k^T]D$$
$$=D[\lambda_1 J+(\lambda_2-\lambda_1)U_2\Omega_2U_2^T+\cdots+(\lambda_k-\lambda_1)U_k\Omega_kU_k^T]D, \tag{3.16}$$

where $U_2, \cdots, U_k$ satisfy

$$U_i^TU_i=I^{(r_i)}, \quad i=2, \cdots, k, \tag{3.17}$$
$$U_i^TJU_j=0, \quad 2\leqslant i<j\leqslant k \tag{3.18}$$

and

$$U_i^TJU_i\Omega_i=I^{(r_i)}, \quad i=2, \cdots, k. \tag{3.19}$$

Let

$$a=(a_{11}, a_{12}, \cdots, a_{1n}, a_{22}, a_{23}, \cdots, a_{2n}, \cdots, a_{nn})^T\in \mathbb{R}^{\frac{n(n+1)}{2}}, \tag{3.20}$$
$$\lambda=(\lambda_1, \cdots, \lambda_k)^T\in \mathbb{R}^k, \tag{3.21}$$
$$d=(d_1, \cdots, d_{n'})^T\in \mathbb{R}^{n'}, \tag{3.22}$$
$$\omega=(\omega_{21}, \cdots, \omega_{2,r_2}, \omega_{31}, \cdots, \omega_{3,r_3}, \cdots, \omega_{k1}, \cdots, \omega_{k,r_k})^T\in \mathbb{R}^{n'-r_1}, \tag{3.23}$$
$$(U_2, \cdots, U_k)=(u_{r_1+1}, \cdots, u_{n'}), \quad u=(u_{r_1+1}^T, \cdots, u_{n'}^T)^T\in \mathbb{R}^{n'(n'-r_1)} \tag{3.24}$$

and

$$\mathscr{E}=S\mathbb{R}^{n\times n}\times \mathbb{R}^k\times \mathbb{R}^{n'}\times \mathbb{R}^{n'-r_1}\times \mathbb{R}^{n'(n'-r_1)}.$$

Because of the relations (3.16)—(3.19) we define differentiable real-valued functions $g_{ij}$ $(1\leqslant i, j\leqslant n')$, $h_{ij}$ $(1\leqslant i, j\leqslant n'-r_1)$ and $l_{ij}$ $(1\leqslant j\leqslant r_i, 2\leqslant i\leqslant k)$ in the Euclidean space $\mathscr{E}$ as follows:

$$(g_{ij})=A'-D[\lambda_1 J+(\lambda_2-\lambda_1)U_2\Omega_2U_2^T+\cdots+(\lambda_k-\lambda_1)U_k\Omega_kU_k^T]D$$

$$h_{ij}=\begin{cases} u_{r_1+i}^Tu_{r_1+i}-1, & 1\leqslant i\leqslant n'-r_1, \\ u_{r_1+i}^TJu_{r_1+j}, & 1\leqslant i, j\leqslant n'-r_1, i\neq j,\end{cases}$$

$$l_{ij}=u_{r_1+\cdots+r_{i-1}+j}^TJu_{r_1+\cdots+r_{i-1}+j}\omega_{ij}-1, \quad 1\leqslant j\leqslant r_i, 2\leqslant i\leqslant k.$$

Then we set

$$g=(g_{11}, g_{12}, \cdots, g_{1n'}, g_{22}, g_{23}, \cdots, g_{2n'}, \cdots, g_{n'n'}),$$
$$h=(h_{11}, \cdots, h_{1,n'-r_1}, h_{22}, \cdots, h_{2,n'-r_1}, \cdots, h_{n'-r_1,n'-r_1}),$$
$$l=(l_{21}, \cdots, l_{2,r_2}, l_{31}, \cdots, l_{3,r_3}, \cdots, l_{k1}, \cdots, l_{k,r_k})$$

and

$$f=(g, l, h).$$

Let $\mathfrak{M}_{1,2,\cdots,n'}\subset\mathscr{E}$ be the set of points $X=\{A, \lambda, d, \omega, u\}\in\mathscr{E}$ such that $f(X)=0$. With each point $X=\{A, \lambda, d, \omega, u\}\in\mathscr{E}$ associates a vector

$$x=(a^T, \lambda^T, d^T, \omega^T, u^T)^T\in \mathbb{R}^{\frac{n(n+1)}{2}+k+2n'-r_1+n'(n'-r_1)},$$

where $A$, $a$, $\lambda$, $d$, $\omega$, $u$ are represented by (3.1) and (3.20)—(3.24). It is easy to see that

$$\frac{\partial f}{\partial x}=\begin{pmatrix}\frac{\partial g}{\partial a} & \frac{\partial l}{\partial a} & \frac{\partial h}{\partial a}\\ \frac{\partial g}{\partial \lambda} & \frac{\partial l}{\partial \lambda} & \frac{\partial h}{\partial \lambda}\\ \frac{\partial g}{\partial d} & \frac{\partial l}{\partial d} & \frac{\partial h}{\partial d}\\ \frac{\partial g}{\partial \omega} & \frac{\partial l}{\partial \omega} & \frac{\partial h}{\partial \omega}\\ \frac{\partial g}{\partial u} & \frac{\partial l}{\partial u} & \frac{\partial h}{\partial u}\end{pmatrix}=\begin{pmatrix}G & 0 & 0\\ * & 0 & 0\\ * & 0 & 0\\ * & L & 0\\ * & * & H\end{pmatrix},$$

where

$$G=\frac{\partial g}{\partial a}=\begin{array}{c}\left(\begin{array}{cccccc}I^{(n')} & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & I^{(n'-1)} & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 1\\ 0 & 0 & 0 & \cdots & 0 & 0\\ & & & 0 & & \end{array}\right)\end{array}\begin{array}{l}\\ n-n'\\ \\ n-n'\\ \\ \\ n-n'\\ \frac{(n-n')(n-n'+1)}{2}\end{array}$$

$$L=\frac{\partial l}{\partial \omega}=\mathrm{diag}(u_{r_1+1}^T J u_{r_1+1},\ u_{r_1+2}^T J u_{r_1+2},\ \cdots,\ u_{n'}^T J u_{n'})$$

and

$$H=\frac{\partial h}{\partial u}=\begin{pmatrix}2u_{r_1+1} & Ju_{r_1+2} & \cdots & Ju_{n'} & 0 & 0 & \cdots & 0 & \cdots & 0\\ & Ju_{r_1+1} & & & 2u_{r_1+2} & Ju_{r_1+3} & \cdots & Ju_{n'} & \cdots & 0\\ & & \ddots & 0 & & Ju_{r_1+2} & \ddots & 0 & & \vdots\\ 0 & & & Ju_{r_1+1} & 0 & & & Ju_{r_1+2} & \cdots & 2u_{n'}\end{pmatrix}.$$

Obviously

$$G^TG=I^{\left(\frac{n'(n'+1)}{2}\right)},\quad \det L=\prod_{i=2}^{k}\det R_i^{-1}\neq 0.$$

Besides, utilizing the relations (3.17)—(3.19) from

$$Hv=0,\quad v\in \mathbb{R}^{(n'-r_1)(n'-r_1+1)/2},$$

we can deduce $v=0$. Therefore

$$\mathrm{rank}(G)=\frac{n'(n'+1)}{2},\quad \mathrm{rank}(L)=n'-r_1,\quad \mathrm{rank}(H)=\frac{(n'-r_1)(n'-r_1+1)}{2}.$$

Hence for each point $X\in\mathfrak{M}_{1,2,\cdots,n'}$ the matrix $\frac{\partial f}{\partial x}$ in which each partial derivative is evaluated at $X$ has

$$\operatorname{rank}\left(\frac{\partial f}{\partial x}\right)=\frac{n'(n'+1)}{2}+n'-r_1+\frac{(n'-r_1)(n'-r_1+1)}{2}.$$

By Lemma 3.1, it follows from

$$\dim(\mathscr{E})=\frac{n(n+1)}{2}+k+2n'-r_1+n'(n'-r_1)$$

that $\mathfrak{M}_{1,2,\cdots,n'}$ is a submanifold of $\mathscr{E}$ with

$$\dim(\mathfrak{M}_{1,2,\cdots,n'})=\dim(\mathscr{E})-\operatorname{rank}\left(\frac{\partial f}{\partial x}\right)=\frac{n(n+1)}{2}+k-\frac{r_1(r_1-1)}{2}. \tag{3.25}$$

Let

$$\mathscr{K}=S\mathsf{R}^{n\times n}\times\mathsf{R}^k,$$

and we define a differentiable mapping $F$ of $\mathfrak{M}_{1,2,\cdots,n'}\to\mathscr{K}$:

$$F(X)=\{A,\lambda\}\in\mathscr{K}\text{ for }X=\{A,\lambda,d,\omega,u\}\in\mathfrak{M}_{1,2,\cdots,n'}$$

and write

$$\mathfrak{M}'_{1,2,\cdots,n'}=F(\mathfrak{M}_{1,2,\cdots,n'}).$$

Observe that

$$\dim(\mathscr{K})=\frac{n(n+1)}{2}+k,$$

and from

$$\dim(\mathscr{K})-\dim(\mathfrak{M}_{1,2,\cdots,n'})=\frac{r_1(r_1-1)}{2}$$

it follows that $\dim(\mathfrak{M}_{1,2,\cdots,n'})<\dim(\mathscr{K})$ if

$$r_1>1. \tag{3.26}$$

Therefore by Lemma 3.2 the set $\mathfrak{M}'_{1,2,\cdots,n'}$ has measure zero under the assumption of (3.26).

Let

$$\mathscr{K}_+=S\mathsf{R}_+^{n\times n}\times\mathsf{R}^k,$$

and let $\mathscr{L}_{1,2,\cdots,n'}$ denote the set of points $X^{(1)}=\{A,\lambda\}\in\mathscr{K}_+$ for which there exist non-zero real numbers $c_1,\cdots,c_{n'}$ such that the matrix $\operatorname{diag}(c_1,\cdots,c_{n'},0^{(r_0)})A$ has a zero eigenvalue of multiplicity $r_0$ and non-zero eigenvalues $\lambda_1,\cdots,\lambda_k$ of multiplicity $r_1,\cdots,r_k$, respectively. Observe that for any point $X^{(1)}=\{A,\lambda\}\in\mathscr{L}_{1,2,\cdots,n'}$ there exist $d\in\mathsf{R}^{n'}$, $\omega\in\mathsf{R}^{n'-r_1}$ and $u\in\mathsf{R}^{n'(n'-r_1)}$ such that the point $X=\{A,\lambda,d,\omega,u\}\in\mathfrak{M}_{1,2,\cdots,n'}$, for this reason $\mathscr{L}_{1,2,\cdots,n'}\subset\mathfrak{M}'_{1,2,\cdots,n'}$, and thus under the assumption of (3.26) the set $\mathscr{L}_{1,2,\cdots,n'}$ has measure zero in the space $\mathscr{K}$.

For arbitrary $n'$ indexes $i_1,i_2,\cdots,i_{n'}$ from the set $\{1,2,\cdots,n\}$, $i_1<i_2<\cdots<i_{n'}$, we consider the case of

$$A'=\begin{pmatrix}a_{i_1,i_1}&\cdots&a_{i_1,i_{n'}}\\ \vdots&&\vdots\\ a_{i_{n'},i_1}&\cdots&a_{i_{n'},i_{n'}}\end{pmatrix}.$$

In much the same way as above we determine a submanifold $\mathfrak{M}_{i_1,i_2,\cdots,i_{n'}}$ of $\mathscr{E}$ and a subset $\mathscr{L}_{i_1,i_2,\cdots,i_{n'}}$ of $\mathscr{K}_+$ and we can prove that the set $\mathscr{L}_{i_1,i_2,\cdots,i_{n'}}$ has measure zero in the space $\mathscr{K}$.

Let $\mathscr{L}$ denote the set of points $X^{(1)}=\{A,\lambda\}\in\mathscr{K}_+$ at which Problem M-1 is solvable. Since

$$\mathscr{L}=\bigcup_{1\le i_1<i_2<\cdots<i_{n'}\le n}\mathscr{L}_{(i_1,i_2,\cdots,i_{n'})}$$

the set $\mathscr{L}$ has measure zero in the space $\mathscr{K}$. This means that Problem M-1 is u.s.a.e. if the condition (2.1) is fulfilled.

2) Suppose that $k$ is an arbitrarily fixed index in the set $\{1, 2, \cdots, n\}$. Let

$$\mathscr{K}=S\mathbb{R}^{n\times n}\times\mathbb{R}^k,$$

$$\mathscr{K}_+=S\mathbb{R}_+^{n\times n}\times\mathbb{R}^k,$$

$$d(\lambda)=\min_{\substack{1\le i,j\le k\\ i\ne j}}|\lambda_i-\lambda_j|\quad\forall\lambda=(\lambda_1,\cdots,\lambda_k)^T\in\mathbb{R}^k,$$

$$\mathscr{K}_+^*=\left\{\{A,\lambda\}\in\mathscr{K}_+:\ A=\begin{array}{cc}\begin{pmatrix}A_1 & *\\ * & *\end{pmatrix} & \begin{array}{l}k\\ n-k\end{array}\\ \begin{array}{cc}k & n-k\end{array} & \end{array},\quad d(\lambda)>\frac{4k_1(A_1)\|\lambda\|_\infty}{1-k_1(A_1)}>0\right\}$$

and

$$\mathscr{K}_0=\left\{\{A,\lambda\}\in\mathscr{K}_+:\ \lambda=(\lambda_1,\cdots,\lambda_k)^T,\ \prod_{i=1}^k\lambda_i\ne0\right\}.$$

Hadeler (see [4], Satz 4) has proved that if $\{A,\lambda\}\in\mathscr{K}_+^*\cap\mathscr{K}_0$ then Problem M-1 has a solution $C=\mathrm{diag}(c_1,\cdots,c_k,0^{(n-k)})\in\mathbb{R}^{n\times n}$. Observe that $\mathscr{K}_+^*\cap\mathscr{K}_0$ is a nonempty open set of the Euclidean space $\mathscr{K}$ and so meas $(\mathscr{K}_+^*\cap\mathscr{K}_0)>0$ if $r_1=\cdots=r_k=1$; hence inequality (2.1) is not only a sufficient but also a necessary condition for the unsolvability of Problem M-1 a.e. ∎

*Proof of Theorem* 2.2.

1) First observe that if Problem GM-1 is solvable at $A_1,\cdots,A_m\in S\mathbb{R}^{n\times n}$ and $\lambda=(\lambda_1,\cdots,\lambda_k)^T\in\mathbb{R}^k$, then there exist a matrix $U\in\mathbb{O}^{n\times n}$ and a vector $c=(c_1,\cdots,c_m)^T\in\mathbb{R}^m$ such that

$$c_1A_1+\cdots+c_mA_m=U\,\mathrm{diag}(0^{(r_0)},\lambda_1I^{(r_1)},\cdots,\lambda_kI^{(r_k)})U^T,\tag{3.27}$$

where

$$A_t=(a_{ij}^{(t)}),\ t=1,\cdots,m,\ U=(U_0,U_1,\cdots,U_k),\ U_i\in\mathbb{R}^{n\times r_i},\ 0\le i\le k.\tag{3.28}$$

Suppose that $r_j=\max\{r_0,r_1,\cdots,r_k\}$ for some index $j\in\{0,1,\cdots,k\}$. Then

$$c_1A_1+\cdots+c_mA_m=\lambda_jI-\lambda_jU_0U_0^T+\sum_{\substack{i=1\\ i\ne j}}^k(\lambda_i-\lambda_j)U_iU_i^T.\tag{3.29}$$

We write

$$(U_0,U_1,\cdots,U_{j-1},U_{j+1},\cdots,U_k)=(u_1,u_2,\cdots,u_{n-r_j}).$$

Since $U\in\mathbb{O}^{n\times n}$ we have

$$(u_1,u_2,\cdots,u_{n-r_j})^T(u_1,u_2,\cdots,u_{n-r_j})=I^{(n-r_j)}.\tag{3.30}$$

Let

$$a_t=(a_{11}^{(t)},a_{12}^{(t)},\cdots,a_{1n}^{(t)},a_{22}^{(t)},a_{23}^{(t)},\cdots,a_{2n}^{(t)},\cdots,a_{nn}^{(t)})^T\in\mathbb{R}^{\frac{n(n+1)}{2}},\quad 1\le t\le m,\tag{3.31}$$

$$\lambda=(\lambda_1,\cdots,\lambda_k)^T\in\mathbb{R}^k,\qquad c=(c_1,\cdots,c_m)^T\in\mathbb{R}^m,\tag{3.32}$$

$$u=(u_1^T,\cdots,u_{n-r_j}^T)^T\in\mathbb{R}^{n(n-r_j)}\tag{3.33}$$

and

$$\mathscr{E}=\underbrace{S\mathbb{R}^{n\times n}\times\cdots\times S\mathbb{R}^{n\times n}}_{m}\times\mathbb{R}^k\times\mathbb{R}^m\times\mathbb{R}^{n(n-r_j)}.$$

Because of the relations (3.29) and (3.30) we define differentiable real-valued functions $g_{pq}$ $(1\leqslant p,\ q\leqslant n)$ and $h_{pq}$ $(1\leqslant p,\ q\leqslant n-r_j)$ in the Euclidean space $\mathscr{E}$ as follows:

$$(g_{pq})=\sum_{i=1}^m c_iA_i-\lambda_jI+\lambda_jU_0U_0^T-\sum_{\substack{i=1\\ i\neq j}}^k(\lambda_i-\lambda_j)U_iU_i^T,$$

$$(h_{pq})=(u_1,\ \cdots,\ u_{n-r_j})^T(u_1,\ \cdots,\ u_{n-r_j})-I^{(n-r_j)},$$

and then set

$$g=(g_{11},\ g_{12},\ \cdots,\ g_{1n},\ g_{22},\ g_{23},\ \cdots,\ g_{2n},\ \cdots,\ g_{nn}),$$

$$h=(h_{11},\ h_{12},\ \cdots,\ h_{1,n-r_j},\ h_{22},\ h_{23},\ \cdots,\ h_{2,n-r_j},\ \cdots,\ h_{n-r_j,n-r_j})$$

and

$$f=(g,\ h).$$

Let $\mathfrak{M}\subset\mathscr{E}$ be the set of points $X=\{A_1,\ \cdots,\ A_m,\ \lambda,\ c,\ u\}\in\mathscr{E}$ such that $f(X)=0$. With each point $X=\{A_1,\ \cdots,\ A_m,\ \lambda,\ c,\ u\}\in\mathscr{E}$ associates a vector

$$x=(a_1^T,\ \cdots,\ a_m^T,\ \lambda^T,\ c^T,\ u^T)^T\in\mathbb{R}^{\frac{mn(n+1)}{2}+k+m+n(n-r_j)},$$

where $A_1,\ \cdots,\ A_m,\ a_1,\ \cdots,\ a_m,\ \lambda,\ c,\ u$ are represented by (3.28) and (3.31)—(3.33). It is easy to verify that

$$\frac{\partial f}{\partial x}=\begin{pmatrix}\frac{\partial g}{\partial a} & \frac{\partial h}{\partial a}\\ \frac{\partial g}{\partial \lambda} & \frac{\partial h}{\partial \lambda}\\ \frac{\partial g}{\partial c} & \frac{\partial h}{\partial c}\\ \frac{\partial g}{\partial u} & \frac{\partial h}{\partial u}\end{pmatrix}=\begin{pmatrix}G & 0\\ * & 0\\ * & 0\\ * & H\end{pmatrix},$$

where

$$a=(a_1^T,\ \cdots,\ a_m^T)^T\in\mathbb{R}^{\frac{mn(n+1)}{2}},$$

$$G=\frac{\partial g}{\partial a}=(c_1I^{(\frac{n(n+1)}{2})},\ \cdots,\ c_mI^{(\frac{n(n+1)}{2})})^T.$$

and

$$H=\frac{\partial h}{\partial u}=\begin{pmatrix}2u_1 & u_2 & \cdots & u_{n-r_j} & 0 & 0 & \cdots & 0 & \cdots & 0\\ & u_1 & & \mathbf{0} & 2u_2 & u_3 & \cdots & u_{n-r_j} & \cdots & 0\\ & & \ddots & & & u_2 & & \mathbf{0} & & \vdots\\ \mathbf{0} & & & \ddots & & & \ddots & & & \vdots\\ & & & & u_1 & & \mathbf{0} & & & 2u_{n-r_j}\end{pmatrix}.$$

From

$$G^TG=(c_1^2+\cdots+c_m^2)I^{(\frac{n(n+1)}{2})}$$

and

$$H^TH=\mathrm{diag}(4,\ 2I^{(n-r_j-1)},\ 4,\ 2I^{(n-r_j-2)},\ \cdots,\ 4,\ 2,\ 4)$$

we see that

$$\mathrm{rank}(G)=\frac{n(n+1)}{2},\quad \mathrm{rank}(H)=\frac{(n-r_j)(n-r_j+1)}{2}.$$

Hence for each point $X\in\mathfrak{M}$ the matrix $\frac{\partial f}{\partial x}$ in which each partial derivative is evaluated at $X$ has

$$\mathrm{rank}\left(\frac{\partial f}{\partial x}\right)=\frac{n(n+1)}{2}+\frac{(n-r_j)(n-r_j+1)}{2}.$$

By Lemma 3.1, it follows from

$$\dim(\mathscr{E})=\frac{mn(n+1)}{2}+k+m+n(n-r_j)$$

that $\mathfrak{M}$ is a submanifold of $\mathscr{E}$ with

$$\begin{aligned}\dim(\mathfrak{M})&=\dim(\mathscr{E})-\mathrm{rank}\left(\frac{\partial f}{\partial x}\right)\\&=\frac{n(m-1)(n+1)}{2}+k+m+\frac{(n-r_j)(n+r_j-1)}{2}.\end{aligned}\tag{3.34}$$

Let

$$\mathscr{X}=\underbrace{S\mathbf{R}^{n\times n}\times\cdots\times S\mathbf{R}^{n\times n}}_{m}\times\mathbf{R}^k,$$

and let $\mathscr{L}$ denote the set of points $X^{(1)}=\{A_1,\ \cdots,\ A_m,\ \lambda\}\in\mathscr{X}$ at which Problem GM-1 is solvable. We define a differentiable mapping $F$ of $\mathfrak{M}\to\mathscr{X}$:

$$F(X)=\{A_1,\ \cdots,\ A_m,\ \lambda\}\in\mathscr{X}\ \text{for}\ X=\{A_1,\ \cdots,\ A_m,\ \lambda,\ c,\ u\}\in\mathfrak{M},$$

and write

$$\mathfrak{M}'=F(\mathfrak{M}).$$

Observe that

$$\dim(\mathscr{X})=\frac{mn(n+1)}{2}+k,$$

and from

$$\dim(\mathscr{X})-\dim(\mathfrak{M})=n-m+\frac{r_j(r_j-1)}{2}$$

it follows that $\dim(\mathfrak{M})<\dim(\mathscr{X})$ if

$$n-m+\frac{r_j(r_j-1)}{2}>0.\tag{3.35}$$

Therefore by Lemma 3.2 the set $\mathfrak{M}'$ has measure zero under the assumption of (3.35). Observe that for any point $X^{(1)}=\{A_1,\ \cdots,\ A_m,\ \lambda\}\in\mathscr{L}$ there exist $c\in\mathbf{R}^m$ and $u\in\mathbf{R}^{n(n-r_j)}$ such that the point $X=\{A_1,\ \cdots,\ A_m,\ \lambda,\ c,\ u\}\in\mathfrak{M}$; hence $\mathscr{L}\subset\mathfrak{M}'$, and thus the set $\mathscr{L}$ has measure zero in the space $\mathscr{X}$. This means that Problem GM-1 is u.s.a.e. if condition (2.2) is fulfilled.

2) We consider the case of $m=n$. Let

$$\mathscr{X}=\underbrace{S\mathbf{R}^{n\times n}\times\cdots\times S\mathbf{R}^{n\times n}}_{n}\times\mathbf{R}^n$$

and

$$\mathscr{K}_s=\{\{A_1, \cdots, A_n, \lambda\}\in\mathscr{K}: A_t=(a_{ij}^{(t)}),\ 1<a_{ii}^{(i)}<2,\ |a_{ii}^{(t)}|<\varepsilon,\ i\neq t,\ 1\leqslant i,\ t\leqslant n\},$$

where $\varepsilon$ is a fixed positive number satisfying $\varepsilon\ll\frac{1}{n-1}$. Obviously, $\mathscr{K}_s$ is an open set of the Euclidean space $\mathscr{K}$, and the matrix

$$E=(a_{ii}^{(j)})_{i,j=1,\cdots,n}$$

is nonsingular provided $\{A_1, \cdots, A_n, \lambda\}\in\mathscr{K}_s$. Let

$$E^{-1}=(l'_{ij}),\ \mu=\sup_{\substack{\{A_1,\cdots,A_n,\lambda\}\in\mathscr{K}_s\\ 1\leqslant i,j\leqslant n}}|l'_{ij}|,$$

$$\tilde{A}_t=\sum_{i=1}^{n}l'_{it}A_i,\quad t=1, \cdots, n$$

and

$$S=\sum_{t=1}^{n}|A_t|,\ \tilde{S}=\sum_{t=1}^{n}|\tilde{A}_t|,\quad \forall\{A_1, \cdots, A_n, \lambda\}\in\mathscr{K}_s,$$

$$d(\lambda)=\min_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}}|\lambda_i-\lambda_j|,\quad \forall\lambda=(\lambda_1, \cdots, \lambda_n)^T\in\mathbb{R}^n$$

and

$$\mathscr{K}^*=\{\{A_1, \cdots, A_n, \lambda\}\in\mathscr{K}_s: d(\lambda)>2n\mu\|\lambda\|_\infty k_2(S)\{[3+n^2\mu^2k_2(S)^2]^{\frac{1}{2}}+n\mu k_2(S)\}\}.$$

From

$$k_2(\tilde{S})\leqslant n\mu k_2(S),\quad \forall\{A_1, \cdots, A_n, \lambda\}\in\mathscr{K}_s$$

we see that if $r=1$ and $\{A_1, \cdots, A_n, \lambda\}\in\mathscr{K}^*$ then by Theorem 6 of [2] Problem GM-1 has a solution $c=(c_1, \cdots, c_n)^T\in\mathbb{R}^n$. Observe that $\mathscr{K}^*$ is a nonempty open set of $\mathscr{K}$ and so meas $\mathscr{K}^*>0$ if $r=1$. Hence in the case of $m=n$ the inequality $r>1$ is not only a sufficient but also a necessary condition for the unsolvability of Problem GM-1 a.e. ∎

*Proof of Theorem* 2.3.

1) Suppose that Problem M-2 is solvable at

$$A=(a_{ij})\in\mathbb{R}^{n\times n},\quad \lambda=(\lambda_1, \cdots, \lambda_k)^T\in\mathbb{R}^k. \tag{3.36}$$

Then there exist an $n'$-dimensional principal submatrix $A'$ of $A$ and a real $n'\times n'$ nonsingular diagonal matrix $C'=\mathrm{diag}(c_1, \cdots, c_{n'})$ such that the matrix $C'A'$ has eigenvalues $\lambda_1, \cdots, \lambda_k$ of multiplicity $r_1, \cdots, r_k$ respectively, in which $n'=r_1+\cdots+r_k$. By hypothesis the matrix $CA$ is diagonalizable. From this the matrix $C'A'$ is diagonalizable too; thus there exist nonsingular matrices $Y$, $Z\in\mathbb{R}^{n'\times n'}$ and nonsingular diagonal matrices

$$D=C'^{-1}=\mathrm{diag}(d_1, \cdots, d_{n'}),\quad \Lambda=\mathrm{diag}(\lambda_1I^{(r_1)}, \cdots, \lambda_kI^{(r_k)}) \tag{3.37}$$

such that

$$A'=DY\Lambda Z^T, \tag{3.38}$$

where

$$Y=(y_1, \cdots, y_{n'}),\quad Z=(z_1, \cdots, z_{n'}) \tag{3.39}$$

and

$$Z^TY=I^{(n')}. \tag{3.40}$$

First we consider the case of

$$A'=\begin{pmatrix} a_{11} & \cdots & a_{1n'} \\ \vdots & & \vdots \\ a_{n'1} & \cdots & a_{n'n'} \end{pmatrix}. \tag{3.41}$$

Without loss of generality we may assume that

$$r_1=\max\{r_1, \cdots, r_k\}$$

and

$$y_i^T y_i=1, \quad i=1, \cdots, n'. \tag{3.42}$$

Then the relations (3.38), (3.40) and (3.42) show that

$$A'=D[\lambda_1 I+(y_{r_1+1}, \cdots, y_{n'})\operatorname{diag}((\lambda_2-\lambda_1)I^{(r_2)}, \cdots, (\lambda_k-\lambda_1)I^{(r_k)})(z_{r_1+1}, \cdots, z_{n'})^T] \tag{3.43}$$

and

$$(z_{r_1+1}, \cdots, z_{n'})^T(y_{r_1+1}, \cdots, y_{n'})=I^{(n'-r_1)}, \quad y_i^T y_i=1, \quad i=r_1+1, \cdots, n'. \tag{3.44}$$

Let

$$a=(a_{11}, a_{12}, \cdots, a_{1n}, a_{21}, a_{22}, \cdots, a_{2n}, \cdots, a_{n1}, a_{n2}, \cdots, a_{nn})^T\in \mathbb{R}^{n^2}, \tag{3.45}$$

$$\lambda=(\lambda_1, \cdots, \lambda_k)^T\in \mathbb{R}^k, \quad d=(d_1, \cdots, d_{n'})^T\in \mathbb{R}^{n'}, \tag{3.46}$$

$$y=(y_{r_1+1}^T, \cdots, y_{n'}^T)^T, \quad z=(z_{r_1+1}^T, \cdots, z_{n'}^T)^T\in \mathbb{R}^{n'(n'-r_1)} \tag{3.47}$$

and

$$\mathscr{E}=\mathbb{R}^{n\times n}\times\mathbb{R}^k\times\mathbb{R}^{n'}\times\mathbb{R}^{n'(n'-r_1)}\times\mathbb{R}^{n'(n'-r_1)}.$$

Because of the relations (3.43) and (3.44) we define differentiable real-valued functions $g_{ij}$ $(1\leqslant i, j\leqslant n')$, $h_{ij}$ $(1\leqslant i, j\leqslant n'-r_1)$ and $l_i$ $(1\leqslant i\leqslant n'-r_1)$ in the Euclidean space $\mathscr{E}$ as follows:

$$\begin{aligned}(g_{ij})&=A'-D[\lambda_1 I\\ &\quad+(y_{r_1+1}, \cdots, y_{n'})\operatorname{diag}((\lambda_2-\lambda_1)I^{(r_2)}, \cdots, (\lambda_k-\lambda_1)I^{(r_k)})(z_{r_1+1}, \cdots, z_{n'})^T],\\ (h_{ij})&=(z_{r_1+1}, \cdots, z_{n'})^T(y_{r_1+1}, \cdots, y_{n'})-I^{(n'-r_1)},\\ l_i&=y_{r_1+i}^T y_{r_1+i}-1, \quad i=1, \cdots, n'-r_1,\end{aligned}$$

and then set

$$\begin{aligned}g&=(g_{11}, g_{12}, \cdots, g_{1n'}, g_{21}, g_{22}, \cdots, g_{2n'}, \cdots, g_{n'1}, g_{n'2}, \cdots, g_{n'n'}),\\ h&=(h_{11}, h_{12}, \cdots, h_{1,n'-r_1}, h_{21}, h_{22}, \cdots, h_{2,n'-r_1}, \cdots, h_{n'-r_1,1},\\ &\qquad h_{n'-r_1,2}, \cdots, h_{n'-r_1,n'-r_1}),\\ l&=(l_1, \cdots, l_{n'-r_1})\end{aligned}$$

and

$$f=(g, h, l).$$

Let $\mathfrak{M}_{1,2,\cdots,n'}\subset\mathscr{E}$ be the set of points $X=\{A, \lambda, d, y, z\}\in\mathscr{E}$ such that $f(X)=0$. With each point $X=\{A, \lambda, d, y, z\}\in\mathscr{E}$ associates a vector

$$x=(a^T, \lambda^T, d^T, y^T, z^T)^T\in \mathbb{R}^{n^2+k+n'+2n'(n'-r_1)}.$$

Here $A$, $a$, $\lambda$, $d$, $y$, $z$ are represented by (3.36) and (3.45)—(3.47). It is easy to verify that

$$\frac{\partial f}{\partial x}=\begin{pmatrix}\frac{\partial g}{\partial a} & \frac{\partial h}{\partial a} & \frac{\partial l}{\partial a}\\ \frac{\partial g}{\partial \lambda} & \frac{\partial h}{\partial \lambda} & \frac{\partial l}{\partial \lambda}\\ \frac{\partial g}{\partial d} & \frac{\partial h}{\partial d} & \frac{\partial l}{\partial d}\\ \frac{\partial g}{\partial y} & \frac{\partial h}{\partial y} & \frac{\partial l}{\partial y}\\ \frac{\partial g}{\partial z} & \frac{\partial h}{\partial z} & \frac{\partial l}{\partial z}\end{pmatrix}=\begin{pmatrix}G & 0 & 0\\ * & 0 & 0\\ * & 0 & 0\\ * & H_y & L_y\\ * & H_z & 0\end{pmatrix},$$

where

$$G=\frac{\partial g}{\partial a}=\begin{array}{c@{}l}\left(\begin{array}{cccccc} I^{(n')} & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & I^{(n')} & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & I^{(n')} & 0\\ 0 & 0 & 0 & \cdots & 0 & 0\\ & & & \mathbf{0} & & \end{array}\right) & \begin{array}{l} \\ n-n'\\ \\ n-n'\\ \\ \\ n-n'\\ n(n-n')\end{array}\end{array},\quad \operatorname{rank}(G)=n'^2,$$

$$H_y=\frac{\partial h}{\partial y}=\begin{pmatrix} z_{r_1+1} & & & z_{r_1+2} & & & \cdots & z_{n'} & & \\ & \ddots & \mathbf{0} & & \ddots & \mathbf{0} & \cdots & & \ddots & \mathbf{0}\\ \mathbf{0} & & & \mathbf{0} & & & \cdots & \mathbf{0} & & \\ & & z_{r_1+1} & & & z_{r_1+2} & \cdots & & & z_{n'}\end{pmatrix}, \tag{3.48}$$

$$H_z=\frac{\partial h}{\partial z}=\begin{pmatrix} y_{r_1+1} & y_{r_1+2} & \cdots & y_{n'} & 0 & 0 & \cdots & 0 & \cdots & & & \\ & & & & y_{r_1+1} & y_{r_1+2} & \cdots & y_{n'} & \cdots & & \mathbf{0} & \\ & \mathbf{0} & & & & \mathbf{0} & & & \cdots & & & \\ & & & & & & & & \cdots & y_{r_1+1} & y_{r_1+2} & \cdots\ y_{n'}\end{pmatrix} \tag{3.49}$$

and

$$L_y=\frac{\partial l}{\partial y}=2\begin{pmatrix} y_{r_1+1} & & \\ & y_{r_1+2} & \\ & & \ddots \\ & & & y_{n'}\end{pmatrix}. \tag{3.50}$$

Let

$$T=\begin{pmatrix} H_y & L_y\\ H_z & 0\end{pmatrix}\in \mathbb{R}^{2n'(n'-r_1)\times(n'-r_1)(n'-r_1+1)}.$$

From $Tw=0$ for any vector $w\in\mathbb{R}^{(n'-r_1)(n'-r_1+1)}$ one can deduce $w=0$. Therefore

$$\operatorname{rank}(T)=(n'-r_1)(n'-r_1+1).$$

Hence for each point $X\in\mathfrak{M}_{1,2,\cdots,n'}$ the matrix $\frac{\partial f}{\partial x}$ in which each partial derivative is evaluated at $X$ has

$$\operatorname{rank}\left(\frac{\partial f}{\partial x}\right)=n'^2+(n'-r_1)(n'-r_1+1).$$

By Lemma 3.1, it follows from

$$\dim(\mathscr{E})=n^2+k+n'+2n(n'-r_1)$$

that $\mathfrak{M}_{1,2,\cdots,n'}$ is a submanifold of $\mathscr{E}$ with

$$\dim(\mathfrak{M}_{1,2,\cdots,n'})=\dim(\mathscr{E})-\operatorname{rank}\left(\frac{\partial f}{\partial x}\right)=n^2+k-r_1(r_1-1). \tag{3.51}$$

Let

$$\mathscr{X}=\mathbf{R}^{n\times n}\times\mathbf{R}^k,$$

and let $\mathscr{L}_{1,2,\cdots,n'}$ denote the set of points $X^{(1)}=\{A,\lambda\}\in\mathscr{X}$ for which there exist non-zero real numbers $c_1, c_2, \cdots, c_{n'}$ such that the matrix $\operatorname{diag}(c_1,\cdots,c_{n'},0^{(r_0)})A$ is diagonalizable and has a zero eigenvalue of multiplicity $r_0$ and non-zero eigenvalues $\lambda_1,\cdots,\lambda_k$ of multiplicity $r_1,\cdots,r_k$ respectively. Then we define a differentiable mapping $F$ of $\mathfrak{M}_{1,2,\cdots,n'}\to\mathscr{X}$:

$$F(X)\triangleq\{A,\lambda\}\in\mathscr{X} \text{ for } X=\{A,\lambda,d,y,z\}\in\mathfrak{M}_{1,2,\cdots,n'}$$

and write

$$\mathfrak{M}'_{1,2,\cdots,n'}=F(\mathfrak{M}_{1,2,\cdots,n'}).$$

Observe that

$$\dim(\mathscr{X})=n^2+k,$$

and from

$$\dim(\mathscr{X})-\dim(\mathfrak{M}_{1,2,\cdots,n'})=r_1(r_1-1)$$

it follows that $\dim(\mathfrak{M}_{1,2,\cdots,n'})<\dim(\mathscr{X})$ if

$$r_1>1. \tag{3.52}$$

Therefore by Lemma 3.2 the set $\mathfrak{M}'_{1,2,\cdots,n'}$ has measure zero under the assumption of (3.52). Observe that for any point $X^{(1)}=\{A,\lambda\}\in\mathscr{L}_{1,2,\cdots,n'}$ there exist $d\in\mathbf{R}^{n'}$ and $y, z\in\mathbf{R}^{n(n'-r_1)}$ such that the point $X=\{A,\lambda,d,y,z\}\in\mathfrak{M}_{1,2,\cdots,n'}$. For this reason $\mathscr{L}_{1,2,\cdots,n'}\subset\mathfrak{M}'_{1,2,\cdots,n'}$, and thus the set $\mathscr{L}_{1,2,\cdots,n'}$ has measure zero in the space $\mathscr{X}$.

For arbitrary $n'$ indexes $i_1, i_2, \cdots, i_{n'}$ from the set $\{1, 2, \cdots, n\}$, $i_1<i_2<\cdots<i_{n'}$, we consider the case of

$$A'=\begin{pmatrix} a_{i_1,i_1} & \cdots & a_{i_1,i_{n'}} \\ \vdots & & \vdots \\ a_{i_{n'},i_1} & \cdots & a_{i_{n'},i_{n'}} \end{pmatrix}.$$

In much the same way as above we determine a submanifold $\mathfrak{M}_{i_1,i_2,\cdots,i_{n'}}$ of $\mathscr{E}$ and a subset $\mathscr{L}_{i_1,i_2,\cdots,i_{n'}}$ of $\mathscr{X}$, and we can prove that the set $\mathscr{L}_{i_1,i_2,\cdots,i_{n'}}$ has measure zero in the space $\mathscr{X}$. of

Let $\mathscr{L}$ denote the set of points $X^{(1)}=\{A,\lambda\}\in\mathscr{X}$ at which Problem M-2 is solvable. Since

$$\mathscr{L}=\bigcup_{1\le i_1<i_2<\cdots<i_{n'}\le n}\mathscr{L}_{i_1,i_2,\cdots,i_{n'}},$$

the set $\mathscr{L}$ has measure zero in the space $\mathscr{X}$. This means that Problem M-2 is u.s.a.e. if condition (2.3) is fulfilled.

2) Suppose that $k$ is an arbitrarily fixed index in the set $\{1, 2, \cdots, n\}$. Let

$$\mathscr{X}=\mathbf{R}^{n\times n}\times\mathbf{R}^{k}$$

and

$$\mathscr{X}_1=\left\{\{A, \lambda\}\in\mathscr{X}:\ A=\begin{array}{cc} \begin{pmatrix} A_1 & * \\ * & * \end{pmatrix} & \begin{array}{l} k \\ n-k \end{array} \\ \begin{array}{cc} k & n-k \end{array} & \end{array},\ A_1=(a_{ij}),\ a_{ii}>1,\ 1\leqslant i\leqslant k\right\}.$$

Obviously, $\mathscr{X}_1$ is an open set of the Euclidean space $\mathscr{X}$, and the matrix

$$E=\mathrm{diag}(a_{11}, a_{22}, \cdots, a_{kk}, I^{(n-k)})$$

is nonsingular provided $\{A, \lambda\}\in\mathscr{X}_1$. Let

$$\tilde{A}=\mathrm{diag}\left(\frac{1}{a_{11}}, \frac{1}{a_{22}}, \cdots, \frac{1}{a_{kk}}, I^{(n-k)}\right)A=\begin{array}{cc} \begin{pmatrix} \tilde{A}_1 & * \\ * & * \end{pmatrix} & \begin{array}{l} k \\ n-k \end{array} \\ \begin{array}{cc} k & n-k \end{array} & \end{array},\quad \forall A=(a_{ij})\in\mathscr{X}_1,$$

$$d(\lambda)=\min_{\substack{1\leqslant i,j\leqslant k\\ i\neq j}}|\lambda_i-\lambda_j|,\quad \forall\lambda=(\lambda_1, \cdots, \lambda_k)^T\in\mathbf{R}^k,$$

$$\mathscr{X}^*=\left\{\{A, \lambda\}\in\mathscr{X}_1:\ A=\begin{array}{cc} \begin{pmatrix} A_1 & * \\ * & * \end{pmatrix} & \begin{array}{l} k \\ n-k \end{array} \\ \begin{array}{cc} k & n-k \end{array} & \end{array},\ d(\lambda)>\frac{4k_1(A_1)\|\lambda\|_\infty}{1-k_1(A)}>0\right\}$$

and

$$\mathscr{X}_0=\left\{\{A, \lambda\}\in\mathscr{X}_1:\ \lambda=(\lambda_1, \cdots, \lambda_k)^T,\ \prod_{i=1}^{k}\lambda_i\neq 0\right\}.$$

From

$$k_1(\tilde{A}_1)\leqslant k_1(A_1),\quad \forall\{A, \lambda\}\in\mathscr{X}_1,\quad A=\begin{array}{cc} \begin{pmatrix} A_1 & * \\ * & * \end{pmatrix} & \begin{array}{l} k \\ n-k \end{array} \\ \begin{array}{cc} k & n-k \end{array} & \end{array}$$

we see that if $r_1=\cdots=r_k=1$ and $\{A, \lambda\}\in\mathscr{X}^*\cap\mathscr{X}_0$, then according to Remark 3 of [2] Problem M-2 has a solution $C=\mathrm{diag}(c_1, \cdots, c_k, 0)\in\mathbf{R}^{n\times n}$. Observe that $\mathscr{X}^*\cap\mathscr{X}_0$ is a nonempty open set of $\mathscr{X}$ and so $\mathrm{meas}(\mathscr{X}^*\cap\mathscr{X}_0)>0$ if $r_1=\cdots=r_k=1$. Hence inequality (2.3) is not only a sufficient but also a necessary condition for the unsolvability of Problem M-2 a.e. ∎

*Proof of Theorem* 2.4.

1) First observe that if Problem GM-2 is solvable at $A_1, \cdots, A_m\in\mathbf{R}^{n\times n}$ and

$$\lambda=(\lambda_1, \cdots, \lambda_k)^T\in\mathbf{R}^k,$$

then there exist nonsingular matrices $Y, Z\in\mathbf{R}^{n\times n}$ and a vector $c=(c_1, \cdots, c_m)^T\in\mathbf{R}^m$ such that

$$c_1A_1+\cdots+c_mA_m=Y\,\mathrm{diag}(0^{(r_0)}, \lambda_1 I^{(r_1)}, \cdots, \lambda_k I^{(r_k)})Z, \tag{3.53}$$

where [illegible]

$$A_t=(a_{ij}^{(t)}),\quad t=1, \cdots, m, \text{ [illegible]} \tag{3.54}$$

$$Y=(Y_0, Y_1, \cdots, Y_k),\ Z=(Z_0, Z_1, \cdots, Z_k),\ Y_i, Z_i\in \mathbb{R}^{n\times r_i},\ i=0, 1, \cdots, k \tag{3.55}$$

and

$$Z^TY=I^{(n)}. \tag{3.56}$$

Suppose that $r_j=\max\{r_0, r_1, \cdots, r_k\}$ for some index $j\in\{0, 1, \cdots, k\}$. Then

$$c_1A_1+\cdots+c_mA_m=\lambda_jI-\lambda_jY_0Z_0^T+\sum_{\substack{i=1\\ i\neq j}}^{k}(\lambda_i-\lambda_j)Y_iZ_i^T. \tag{3.57}$$

We write

$$(Y_0, Y_1, \cdots, Y_{j-1}, Y_{j+1}, \cdots, Y_k)=(y_1, y_2, \cdots, y_{n-r_j}),$$
$$(Z_0, Z_1, \cdots, Z_{j-1}, Z_{j+1}, \cdots, Z_k)=(z_1, z_2, \cdots, z_{n-r_j}).$$

From (3.56)

$$(z_1, z_2, \cdots, z_{n-r_j})^T(y_1, y_2, \cdots, y_{n-r_j})=I^{(n-r_j)}. \tag{3.58}$$

Besides, we may assume without loss of generality that

$$y_i^Ty_i=1, \quad i=1, 2, \cdots, n-r_j. \tag{3.59}$$

Let

$$a_t=(a_{11}^{(t)}, a_{12}^{(t)}, \cdots, a_{1n}^{(t)}, a_{21}^{(t)}, a_{22}^{(t)}, \cdots, a_{2n}^{(t)}, \cdots, a_{n1}^{(t)}, a_{n2}^{(t)}, \cdots, a_{nn}^{(t)})^T\in\mathbb{R}^{n^2},\ 1\leqslant t\leqslant m, \tag{3.60}$$

$$\lambda=(\lambda_1, \cdots, \lambda_k)^T\in\mathbb{R}^k, \quad c=(c_1, \cdots, c_m)^T\in\mathbb{R}^m, \tag{3.61}$$

$$y=(y_1^T, \cdots, y_{n-r_j}^T)^T, \quad z=(z_1^T, \cdots, z_{n-r_j}^T)^T\in\mathbb{R}^{n(n-r_j)} \tag{3.62}$$

and

$$\mathscr{E}=\underbrace{\mathbb{R}^{n\times n}\times\cdots\times\mathbb{R}^{n\times n}}_{m}\times\mathbb{R}^k\times\mathbb{R}^m\times\mathbb{R}^{n(n-r_j)}.$$

Because of the relations (3.57)—(3.59) we define differentiable real-valued functions $g_{pq}$ $(1\leqslant p, q\leqslant n)$, $h_{pq}(1\leqslant p, q\leqslant n-r_j)$ and $l_i$ $(1\leqslant i\leqslant n-r_j)$ in the Euclidean space $\mathscr{E}$ as follows:

$$(g_{pq})=\sum_{i=1}^{m}c_iA_i-\lambda_jI+\lambda_jY_0Z_0^T-\sum_{\substack{i=1\\ i\neq j}}^{k}(\lambda_i-\lambda_j)Y_iZ_i^T,$$

$$(h_{pq})=(z_1, z_2, \cdots, z_{n-r_j})^T(y_1, y_2, \cdots, y_{n-r_j})-I^{(n-r_j)},$$

$$l_i=y_i^Ty_i-1, \quad i=1, 2, \cdots, n-r_j,$$

and then set

$$g(g_{11}, g_{12}, \cdots, g_{1n}, g_{21}, g_{22}, \cdots, g_{2n}, \cdots, g_{n1}, g_{n2}, \cdots, g_{nn}),$$
$$h=(h_{11}, h_{12}, \cdots, h_{1,n-r_j}, h_{21}, h_{22}, \cdots, h_{2,n-r_j}, \cdots, h_{n-r_j,1}, h_{n-r_j,2}, \cdots, h_{n-r_j,n-r_j}),$$
$$l=(l_1, l_2, \cdots, l_{n-r})$$

and

$$f=(g, h, l).$$

Let $\mathfrak{M}\subset\mathscr{E}$ be the set of points $X=\{A_1, \cdots, A_m, \lambda, c, y, z\}\in\mathscr{E}$ such that $f(X)=0$. With each point $X=\{A_1, \cdots, A_m, \lambda, c, y, z\}\in\mathscr{E}$ associates a vector

$$x=(a_1^T, \cdots, a_m^T, \lambda^T, c^T, y^T, z^T)^T\in\mathbb{R}^{mn^2+k+m+2n(n-r_j)},$$

where $A_1, \cdots, A_m, a_1, \cdots, a_m, \lambda, c, y, z$ are represented by (3.54) and (3.60)—(3.62). It is easy to verify that

$$\frac{\partial f}{\partial x}=\begin{pmatrix} \frac{\partial g}{\partial a} & \frac{\partial h}{\partial a} & \frac{\partial l}{\partial a} \\ \frac{\partial g}{\partial \lambda} & \frac{\partial h}{\partial \lambda} & \frac{\partial l}{\partial \lambda} \\ \frac{\partial g}{\partial c} & \frac{\partial h}{\partial c} & \frac{\partial l}{\partial c} \\ \frac{\partial g}{\partial y} & \frac{\partial h}{\partial y} & \frac{\partial l}{\partial y} \\ \frac{\partial g}{\partial z} & \frac{\partial h}{\partial z} & \frac{\partial l}{\partial z} \end{pmatrix}=\begin{pmatrix} G & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \\ * & H_y & L_y \\ * & H_z & 0 \end{pmatrix},$$

where

$$a=(a_1^T, \cdots, a_m^T)^T \in \mathbb{R}^{mn^2},$$
$$G=(c_1 I^{(n^2)}, \cdots, c_m I^{(n^2)})$$

and $H_y$, $H_z$, $L_y$ are defined as in (3.48)—(3.50). Thus

$$\operatorname{rank}(G)=n^2, \quad \operatorname{rank}\left(\begin{pmatrix} H_y & L_y \\ H_z & 0 \end{pmatrix}\right)=(n-r_j)(n-r_j+1).$$

Hence for each point $X \in \mathfrak{M}$ the matrix $\frac{\partial f}{\partial x}$ in which each partial derivative is evaluated at $X$ has

$$\operatorname{rank}\left(\frac{\partial f}{\partial x}\right)=n^2+(n-r_j)(n-r_j+1).$$

By Lemma 3.1, it follows from

$$\dim(\mathscr{E})=mn^2+k+m+2n(n-r_j)$$

that $\mathfrak{M}$ is a submanifold of $\mathscr{E}$ with

$$\dim(\mathfrak{M})=\dim(\mathscr{E})-\operatorname{rank}\left(\frac{\partial f}{\partial x}\right)=(m-1)n^2+k+m+(n-r_j)(n+r_j-1). \quad (3.63)$$

Let

$$\mathscr{K}=\underbrace{\mathbb{R}^{n\times n}\times\cdots\times\mathbb{R}^{n\times n}}_{m}\times\mathbb{R}^k,$$

and let $\mathscr{L}$ denote the set of points $X^{(1)}=\{A_1, \cdots, A_m, \lambda\}\in\mathscr{K}$ at which Problem GM-2 is solvable. We define a differentiable mapping $F$ of $\mathfrak{M}\to\mathscr{K}$:

$$F(X)=\{A_1, \cdots, A_m, \lambda\}\in\mathscr{K} \quad \text{for} \quad X=\{A_1, \cdots, A_m, \lambda, c, y, z\}\in\mathfrak{M},$$

and write

$$\mathfrak{M}'=F(\mathfrak{M}).$$

Observe that

$$\dim(\mathscr{K})=mn^2+k,$$

and from

$$\dim(\mathscr{K})-\dim(\mathfrak{M})=n-m+r_j(r_j-1)$$

it follows that $\dim(\mathfrak{M})<\dim(\mathscr{K})$ if

$$n-m+r_j(r_j-1)>0. \quad (3.64)$$

Therefore by Lemma 3.2 the set $\mathfrak{M}'$ has measure zero under the assumption of (3.64). Observe that for any point $X^{(1)}=\{A_1, \cdots, A_m, \lambda\}\in\mathscr{L}$ there exist $c\in\mathbb{R}^m$ and

$y, z\in \mathbb{R}^{n(n-r_1)}$ such that the point $X=\{A_1, \cdots, A_m, \lambda, c, y, z\}\in\mathfrak{M}$. Hence $\mathscr{L}\subset\mathfrak{M}'$, and thus the set $\mathscr{L}$ has measure zero in the space $\mathscr{X}$. This means that Problem GM-2 is u.s.a.e. if condition (2.4) is fulfilled.

2) We consider the case of $m=n$. Let

$$\mathscr{X}=\underbrace{\mathbb{R}^{n\times n}\times\cdots\times\mathbb{R}^{n\times n}}_{n}\times\mathbb{R}^n$$

and

$$\mathscr{X}_s=\{\{A_1, \cdots, A_n, \lambda\}\in\mathscr{X}: A_t=(a_{ij}^{(t)}),\ 1<a_{tt}^{(t)}<2,\ |a_{it}^{(t)}|<s,\ i\neq t,\ 1\leqslant i, t\leqslant n\},$$

where $s$ is a fixed positive number satisfying $s\ll\frac{1}{n-1}$. Obviously $\mathscr{X}_s$ is an open set of the Euclidean space $\mathscr{X}$, and the matrix

$$E=(a_{it}^{(j)})_{i,j=1,\cdots,n}$$

is nonsingular provided $\{A_1, \cdots, A_n, \lambda\}\in\mathscr{X}_s$. Let

$$E^{-1}=(l'_{ij}),\quad \mu=\sup_{\substack{\{A_1,\cdots,A_n,\lambda\}\in\mathscr{X}_s\\ 1\leqslant i,j\leqslant n}}|l'_{ij}|,$$

$$\widetilde{A}_t=\sum_{i=1}^{n} l'_{it}A_i,\quad t=1, \cdots, n$$

and

$$S=\sum_{t=1}^{n}|A_t|,\ \widetilde{S}=\sum_{t=1}^{n}|\widetilde{A}_t|,\quad \forall\{A_1, \cdots, A_n, \lambda\}\in\mathscr{X}_s,$$

$$d(\lambda)=\min_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}}|\lambda_i-\lambda_j|,\quad \forall\lambda=(\lambda_1, \cdots, \lambda_n)^T\in\mathbb{R}^n$$

and

$$\mathscr{X}^*=\left\{\{A_1, \cdots, A_n, \lambda\}\in\mathscr{X}_s:\ d(\lambda)>\frac{4n\mu k_1(S)\|\lambda\|_\infty}{1-n\mu k_1(S)}>0\right\}.$$

From

$$k_1(\widetilde{S})\leqslant n\mu k_1(S),\quad \forall\{A_1, \cdots, A_n, \lambda\}\in\mathscr{X}_s$$

we see that if $r=1$ and $\{A_1, \cdots, A_n, \lambda\}\in\mathscr{X}^*$ then by Theorem 1 of [2] Problem GM-2 has a solution $c=(c_1, \cdots, c_n)^T\in\mathbb{R}^n$. Observe that $\mathscr{X}^*$ is a nonempty open set of $\mathscr{X}$ and so meas $\mathscr{X}^*>0$ if $r=1$. Hence in the case of $m=n$ the inequality $r>1$ is not only a sufficient but also a necessary condition for the unsolvability of Problem GM-2 a.e. ∎

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