Vol. 5 No. 3 JOURNAL OF COMPUTATIONAL MATHEMATICS July 1987

STABILITY OF IMPLICIT DIFFERENCE SCHEMES WITH SPACE AND TIME-DEPENDENT COEFFICIENTS*[1)]

YONG WEN-AN (雍稳安) ZHU YOU-LAN (朱幼兰)
(Computing Center, Academia Sinica, Beijing, China)

Abstract

A stability theorem is derived for implicit difference schemes approximating multidimensional initial-value problems for linear hyperbolic systems with variable coefficients, and lots of widely used difference schemes are proved to be stable under the conditions similar to those for the cases of constant coefficients. This theorem is an extension of the stability theorem due to Lax-Nirenberg[3]. The proof is quite simple.

§ 1. Introduction

In the 1960s, stability of difference schemes for initial-value problems for linear hyperbolic systems with variable coefficients was extensively and intensively studied, and some well-known and deep results were obtained, such as Kreiss dissipative theorem[3], Lax-Nirenberg's stability theorem[3].

However, most of the results are only suitable to explicit schemes and some of them are only applicable to the time-independent cases. Also the conditions ensuring the stability of schemes are very strong and hard to be checked. And the proofs are very complicated.

In this paper, combining a skill in [1] with Lax-Nirenberg's theorem for difference operators[2], we obtain a stability theorem. The schemes considered here could be both explicit and implicit, and their coefficients may depend on time variable as well as space ones. The conditions needed are natural and easy to be checked pointwise. The proof is quite simple. This theorem is an extension of Lax-Nirenberg's. As a consequence, we prove that lots of widely used schemes are stable under the conditions similar to those for the cases of constant coefficients. Dissipation and symmetry (conjugacy) are not mentioned.

§ 2. Results

For convenience, we first introduce some notation and state the Lax-Nirenberg theorem for difference operators[2].

Let $P_\alpha(x)$ be $N\times N$ complex matrices with elements depending on variables $x\in R^p$, and $u(x)$ be a complex vector function with N components $\in L^2(R^p)$. The difference operator P_h with a single parameter h(positive real) is defined in the following form:

* Received July 24, 1986.

1) The Project supported by National Natural Science Foundation of China.

$$(P_h u)(x) = \sum_\alpha P_\alpha(x) T^\alpha u(x),$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_p)$, α_i $(i=1, 2, \cdots, p)$ are integers and T^α are shift-operators:

$$T^\alpha u(x) = u(x_1+\alpha_1 h, x_2+\alpha_2 h, \cdots, x_p+\alpha_p h).$$

Define

$$P(x, \xi) = \sum_\alpha P_\alpha(x) e^{i\alpha\cdot\xi} \quad \text{for } \xi \in R^p,$$

$$|P|_{l,m} = \sum_\alpha \sup_{\substack{|\beta|=l \\ x\in R^p}} \|\partial_x^\beta P_\alpha(x)\|_2 \cdot (\alpha)^m,$$

where $\|\cdot\|$ is the spectral norm of matrix, β are multi-indices, β_i $(i=1, 2, \cdots, p)$ are non-negative integers, $|\beta| = \sum_{i=1}^p |\beta_i|$, $(\alpha)^2 = \sum_{i=1}^p \alpha_i^2$, $\partial_x^\beta P_\alpha(x) = \partial_{x_1}^{\beta_1}\partial_{x_2}^{\beta_2}\cdots\partial_{x_p}^{\beta_p} P_\alpha(x)$, and l, m are integers.

The Lax-Nirenberg Theorem. If $P(x, \xi)$ is a non-negative Hermitian matrix for all $x, \xi \in R^p$ and $|P|_{2,0}$, $|P|_{0,2}$ are bounded, then

$$\mathrm{Re}(u, P_h u) \geq -\frac{1}{2} h(C|P|_{0,2} + |P|_{2,0}) \|u\|^2,$$

for all $u \in L^2(R^p)$, where $(\cdot, \cdot)$ is the scalar product in L^2, $\|\cdot\|$ is the corresponding norm, $\mathrm{Re}(u, P_h u)$ is the real part of $(u, P_h u)$ and C is an absolute positive constant.

In this paper, we will discuss the following schemes:

$$\sum_\mu R_\mu(x, t, \Delta) T^\mu u^{n+1}(x) = \sum_\mu S_\mu(x, t, \Delta) T^\mu u^n(x), \tag{*}$$

where the two sides of (*) are similar to the definition of P_h. The difference between them is that the elements of R_μ, S_μ depend on x, as well as on t and Δ (Δ represents time and space meshsizes), and $|\mu|$ are not larger than some constant. Superscript n indicates that vector u depends on time variable $t = n\Delta t$, $n = 0, 1, \cdots, t \leq T$ (constant).

In constant coefficient cases, $\Delta t/h$ is usually a constant. It is more reasonable to assume $\Delta t/h$ to satisfy

$$0 < \mathrm{const}_1 \leq \Delta t/h \leq \mathrm{const}_2 < +\infty,$$

because the coefficients here are variable.

Theorem. *If the following condition* (A) *holds, then the schemes* (*) *are stable with respect to initial-value in the sense of* Lax[3] *with* L^2 *norm, that is, there exists a positive constant* C *such that*

$$\|u^n\| \leq C\|u^0\|, \quad 0 < n\Delta t \leq T, \quad n = 1, 2, \cdots.$$

Condition (A). There exist two positive constants C_1, C_2 and two invertible matrices $M(x, t)$, $G(x, t)$ such that for all $x, \xi \in R^p$, $0 < t \leq T$,

a) $(\sum_\mu MR_\mu(x, t, 0) Ge^{i\mu\xi})^* (\sum_\mu MR_\mu(x, t, 0) Ge^{i\mu\xi}) - (\sum_\mu MS_\mu(x, t, 0) Ge^{i\mu\xi})^* (\sum_\mu MS_\mu(x, t, 0) Ge^{i\mu\xi}) \geq 0$;

b) $(\sum_\mu MR_\mu(x, t, 0) Ge^{i\mu\xi})^* (\sum_\mu MR_\mu(x, t, 0) Ge^{i\mu\xi}) - C_1 I \geq 0$ (I is the unit matrix of order N);

c) $r(x, t, \Delta)$, $s(x, t, \Delta)$ (elements of $R(x, t, \Delta)$, $S(x, t, \Delta)$ respectively) and their main parts $r(x, t, 0)$, $s(x, t, 0)$ satisfy

$$|r(x, t, \Delta) - r(x, t, 0)| \leq C_2 \Delta t, \quad |s(x, t, \Delta) - s(x, t, 0)| \leq C_2 \Delta t,$$

$$|\partial_x^\beta r(x, t, 0)| \leqslant C_2, \quad |\partial_x^\beta s(x, t, 0)| \leqslant C_2, \quad |\beta| \leqslant 2$$

and
$$|r(x, t_1, 0) - r(x, t_2, 0)| \leqslant C_2 |t_1 - t_2|, \quad t_1, t_2 \in (0, T];$$

d) the elements of $M(x, t)$, $G(x, t)$, $G^{-1}(x, t)$ satisfy the same conditions as for $r(x, t, 0)$.

Note. a), b) mean that the corresponding Hermitian matrices there are non-negative definite. For the reason why we introduce such type of conditions, see [1] and later examples.

Proof. Let $H^{n+1} = \int_{R^p} |\sum_\mu M(x, t) R_\mu(x, t, \Delta) T^\mu u^{n+1}(x)|^2 dx$. From c), d), we have

$$H^{n+1} \leqslant \widetilde{C} \|u^{n+1}\|^2, \tag{1}$$

where $\widetilde{C}$ is a positive constant.

Set $v(x, t) = G^{-1}(x, t) u(x, t)$. Obviously, there exist positive constants $\overline{K}_1$, $\overline{K}_2$ such that

$$\overline{K}_1 \|u\|^2 \leqslant \|v\|^2 \leqslant \overline{K}_2 \|u\|^2. \tag{2}$$

For H^{n+1} we have the following expressions:

$$H^{n+1} = (\sum_\mu M R_\mu(x, t, \Delta) G G^{-1} T^\mu u^{n+1}, \sum_\mu M R_\mu(x, t, \Delta) G G^{-1} T^\mu u^{n+1})$$
$$= (\sum_\mu M R_\mu(x, t, \Delta) G (T^\mu (G^{-1}u)^{n+1} + O(\Delta t) T^\mu u^{n+1}),$$
$$\sum_\mu M R_\mu(x, t, \Delta) G (T^\mu (G^{-1}u)^{n+1} + O(\Delta t) T^\mu u^{n+1})$$
$$= \mathrm{Re}(v^{n+1}, \sum_{\mu,\nu} (M R_\mu(x, t, 0) G)^* (M R_\nu(x, t, 0) G) T^{\nu-\mu} v^{n+1}) + O(\Delta t) \|u^{n+1}\|^2$$
$$= C_1 \|v^{n+1}\|^2 + O(\Delta t) \|u^{n+1}\|^2$$
$$+ \mathrm{Re}(v^{n+1}, [\sum_{\mu,\nu} (M R_\mu(x, t, 0) G)^* (M R_\nu(x, t, 0) G) T^{\nu-\mu} - C_1 T^0] v^{n+1}).$$

In the Lax–Nirenberg theorem, we take

$$P_h = \sum_{\mu,\nu} (M R_\mu(x, t, 0) G)^* (M R_\nu(x, t, 0) G) T^{\nu-\mu} - C_1 T^0.$$

According to b), c), d) and (2), there exists a positive constant K_1 such that

$$H^{n+1} \geqslant C_1 \|v^{n+1}\|^2 - K_1 \Delta t \|v^{n+1}\|^2. \tag{3}$$

On the other hand, we have from (*)

$$H^{n+1} - H^n = \mathrm{Re}(v^n, \sum_{\mu,\nu} [(M S_\mu(x, t, 0) G)^* (M S_\nu(x, t, 0) G)$$
$$- (M R_\mu(x, t, 0) G)^* (M R_\nu(x, t, 0) G)] T^{\nu-\mu} v^n) + O(\Delta t) \|u^n\|^2$$

(The Lipschitz continuity of $r(x, t, 0)$ with respect to t is used here).

In the Lax–Nirenberg theorem, we now take $P_h = \sum_{\mu,\nu} [(M R_\mu(x, t, 0) G)^* (M R_\nu(x, t, 0) G) - (M S_\mu(x, t, 0) G)^* (M S_\nu(x, t, 0) G)] T^{\nu-\mu}$. According to a), c), d) and (2), there exists a positive constant K_2 such that

$$H^{n+1} - H^n \leqslant K_2 \Delta t \|v^n\|^2. \tag{4}$$

When Δt is sufficiently small, combining (3) with (4) gives

$$H^{n+1} \leqslant H^n + K_2 \Delta t \|v^n\|^2 \leqslant H^n + [K_2 \Delta t / (C_1 - K_1 \Delta t)] H^n.$$

Let K be a constant larger than $K_2/(C_1 - K_1 \Delta t)$, we have

$$H^{n+1}\leqslant(1+K\Delta t)H^n.$$

Furthermore, $H^n\leqslant(1+K\Delta t)^nH^0\leqslant e^{nK\Delta t}H^0\leqslant e^{KT}H^0$.

Thus, using (1), (2) and (3) we obtain

$$\begin{aligned}\|u^n\|^2&\leqslant(1/\overline{K}_1)\|v^n\|^2\leqslant[1/(\overline{K}_1(C_1-K_1\Delta t))]H^n\\&\leqslant[e^{KT}/(\overline{K}_1(C_1-K_1\Delta t)]H^0\\&\leqslant[\widetilde{C}e^{KT}/(\overline{K}_1(C_1-K_1\Delta t))]\|u^0\|^2.\end{aligned}$$

Taking $C\geqslant\sqrt{\widetilde{C}e^{KT}/(\overline{K}_1(C_1-K_1\Delta t))}$ we have

$$\|u^n\|\leqslant C\|u^0\|.$$

This completes the proof.

Note. Condition b) guarantees that systems of (*) are well-conditioned, which is obvious from (3). Usually, M equals G^{-1}. Therefore, b) becomes $(MG)^*(MG)=I\geqslant C_1I$ for explicit schemes, which always holds if we take $C_1\leqslant1$.

§ 3. Applications

Consider two level difference schemes approximating

$$\frac{\partial U}{\partial t}=\sum_{j=1}^{p}A_j(x,t)\frac{\partial U}{\partial x_j} \tag{**}$$

($t>0$, $x=(x_1, x_2, \cdots, x_n)\in R^p$), which is a hyperbolic system in p dimensions. Supposing that these schemes are written in the form of (*), it is easy to see that for lots of widely used schemes, R_μ, S_μ are some polynomials of $A_j(j=1, 2, \cdots, p)$, such as Lax scheme[4], Lax-Wendroff scheme[4,5] (Richtmyer scheme[4], MacCormack scheme[4]), Keller-Thomée scheme[6], Crank-Nicolson scheme[7], Rusanov scheme[4] and Burstein-Mirin scheme[4], Godynov scheme[4], Harten-Tal-Ezer scheme[8], Abarbanel-Gottlieb-Turkel scheme[4], and so on.

If these A_j are commutable with each other, we can express $R_\mu(x, t, 0)$, $S_\mu(x, t, 0)$ in the following form:

$$\sum_\nu C_\nu A^\nu(x,t)$$

where C_ν are real and $A^\nu=A_1^{\nu_1}A_2^{\nu_2}\cdots A_p^{\nu_p}$. Furthermore, we suppose that there is an invertible matrix $P(x, t)$ such that

$$P(x,t)A_j(x,t)P^{-1}(x,t)=\Lambda_j(x,t)$$

for $j=1, 2, \cdots, p$, where $\Lambda_j(x, t)$ are real diagonal.

Taking $M(x, t)=P(x, t)$, $G(x, t)=P^{-1}(x, t)$, we find that $\sum_\mu MR_\mu(x, t, 0)Ge^{i\mu\xi}$, $\sum_\mu MS_\mu(x, t, 0)Ge^{i\mu\xi}$ are both diagonal. Therefore, it is easy to check a), b).

Example 1. $p=1$, $A_1=A$. Consider the scheme of Harten et al.:

$$\begin{aligned}&U_j^{n+1}+\frac{1}{6}(U_{j+1}^{n+1}-2U_j^{n+1}+U_{j-1}^{n+1})-\frac{\lambda}{4}A_j^{n+1}(U_{j+1}^{n+1}-U_{j-1}^{n+1})\\&=U_j^n+\frac{1}{6}(U_{j+1}^n-2U_j^n+U_{j-1}^n)+\frac{\lambda}{4}A_j^n(U_{j+1}^n-U_{j-1}^n),\quad \lambda=\Delta t/h.\end{aligned}$$

Obviously, $$\sum_\mu R_\mu(x,t,0)e^{i\mu\xi}=\frac{2}{3}I+\frac{\cos\xi}{3}I-\frac{i\lambda A_j^n}{2}\sin\xi,$$

$$\sum_\mu S_\mu(x,\ t,\ 0)e^{i\mu\xi}=\frac{2}{3}I+\frac{\cos\xi}{3}I+\frac{i\lambda A_j^n}{2}\sin\xi.$$

$$(\sum_\mu PR_\mu(x,\ t,\ 0)P^{-1}e^{i\mu\xi})^*(\sum_\mu PR_\mu(x,\ t,\ 0)P^{-1}e^{i\mu\xi})$$
$$-(\sum_\mu PS_\mu(x,\ t,\ 0)P^{-1}e^{i\mu\xi})^*(\sum_\mu PS_\mu(x,\ t,\ 0)P^{-1}e^{i\mu\xi})=0,$$

and
$$(\sum_\mu PR_\mu(x,\ t,\ 0)P^{-1}e^{i\mu\xi})^*(\sum_\mu PR_\mu(x,\ t,\ 0)P^{-1}e^{i\mu\xi})\geqslant\frac{1}{9}I.$$

According to the theorem, the scheme of Harten et al. is absolutely stable.

Example 2. $p=2$, $A_1=A$, $A_2=B$. Considering the Lax-Wendroff scheme[5], we have

$$\sum_\mu R_\mu(x,\ t,\ 0)e^{i\mu\xi}=I,$$

and

$$\sum_\mu S_\mu(x,\ t,\ 0)e^{i\mu\xi}=I+i\lambda(A\sin\xi_1+B\sin\xi_2)$$
$$-\lambda^2\cdot[A^2(1-\cos\xi_1)+B^2(1-\cos\xi_2)+AB\sin\xi_1\sin\xi_2],\quad \lambda=\Delta t/h.$$

By multiplying M, G appropriately, A, B are both transformed into diagonal matrices. Discuss every diagonal element with the method in (5), and we find that stability conditions are

$$\max_{\substack{x_1,x_2,t\\1\leqslant l\leqslant N}}|\lambda\lambda_l(x_1,\ x_2,\ t)|\leqslant\frac{1}{2\sqrt{2}},\quad \max_{\substack{x_1,x_2,t\\1\leqslant l\leqslant N}}|\lambda\mu_l(x_1,\ x_2,\ t)|\leqslant\frac{1}{2\sqrt{2}},$$

where λ_l, μ_l $(l=1,\ 2,\ \cdots,\ N)$ are the eigenvalues of A, B, respectively.

These conditions are similar to those for constant coefficient cases. This result is better than that obtained with Kreiss's dissipative theorem[5].

Example 3. Consider the Keller-Thomée scheme approximating (**),

$$U_\nu^{n+1}-U_\nu^n=\frac{\lambda}{4}\sum_{j=1}^p(A_j)_\nu^{n+\frac{1}{2}}(U_{\nu+e_j}^{n+1}-U_{\nu-e_j}^{n+1}+U_{\nu+e_j}^n-U_{\nu-e_j}^n),$$

where $e_j=(0,\ \cdots,\ 0,\ 1,\ 0,\ \cdots,\ 0)^T(j=1,\ 2,\ \cdots,\ p)$, ν are multi-indices $1,\ \cdots,\ j-1,\ j,\ j+1,\ \cdots,\ p$ indicating the mesh points in the p-dimensional space.

We have

$$\sum_\mu R_\mu(x,\ t,\ 0)e^{i\mu\xi}=I-\frac{i\lambda}{2}\sum_{j=1}^p A_j(x,\ t)\sin\xi_j,$$

and
$$\sum_\mu S_\mu(x,\ t,\ 0)e^{i\mu\xi}=I+\frac{i\lambda}{2}\sum_{j=1}^p A_j(x,\ t)\sin\xi_j.$$

Taking $M=G=I$ (unit matrix), we get

$$(\sum_\mu MR_\mu(x,\ t,\ 0)Ge^{i\mu\xi})^*(\sum_\mu MR_\mu(x,\ t,\ 0)Ge^{i\mu\xi})$$
$$-(\sum_\mu MS_\mu(x,\ t,\ 0)Ge^{i\mu\xi})^*(\sum_\mu MS_\mu(x,\ t,\ 0)Ge^{i\mu\xi})=0,$$

and
$$(\sum_\mu MR_\mu(x,\ t,\ 0)Ge^{i\mu\xi})^*(\sum_\mu MR_\mu(x,\ t,\ 0)Ge^{i\mu\xi})\geqslant I.$$

Thus, the above scheme is absolutely stable (Note. We do not require that those A_j be commutable with each other here).

Similarly, we can obtain the stability conditions for other schemes.

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