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A DIFFERENCE SCHEME FOR A CLASS OF NONLINEAR SCHRÖDINGER EQUATIONS OF HIGH ORDER*

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§1. Introduction

It is well known that the nonlinear equations of Schrödinger type are of great importance to physics and can be used to describe extensive physical phenomena[1,2]. Many authors have discussed the equations of Schrödinger type theoretically and a lot of numerical methods have been presented[3-6].

In this paper, we will consider a class of system of nonlinear Schrödinger equations of high order

$$i\frac{\partial \boldsymbol{u}}{\partial t}+(-1)^m\frac{\partial^m}{\partial x^m}\left(A(x)\frac{\partial^m \boldsymbol{u}}{\partial x^m}\right)+\beta(x)q(|\boldsymbol{u}|^2)\boldsymbol{u}+F(x,t)\boldsymbol{u}=0, \tag{1.1}$$

with the initial condition

$$\boldsymbol{u}|_{t=0}=\boldsymbol{u}_0(x),\quad 0\leqslant x\leqslant D, \tag{1.2}$$

and the homogeneous boundary conditions

$$\left.\frac{\partial^l \boldsymbol{u}}{\partial x^l}\right|_{x=0}=\left.\frac{\partial^l \boldsymbol{u}}{\partial x^l}\right|_{x=D}=0,\quad l=0,\cdots,m-1,\ t\geqslant 0, \tag{1.3}$$

where $i=\sqrt{-1}$, $\boldsymbol{u}=(u_1(x,t),\cdots,u_M(x,t))$ is an unknown M-dimensional vector function, $|\boldsymbol{u}|^2=|u_1|^2+\cdots+|u_M|^2$. Both $F(x,t)=(f_{l,r}(x,t))_{M\times M}$ and $A(x)=\operatorname{diag}(a_1(x),\cdots,a_M(x))$ are given real function matrices which are symmetric, $\beta(x)$ and $q(x)$ are given real functions, and $\boldsymbol{u}_0(x)$ is a given M-dimensional complex vector function satisfying condition (1.3).

Corresponding to the problem (1.1)—(1.3), we present a class of difference schemes which satisfy some important conservation laws of equations (1.1). The convergence and stability of the proposed scheme is derived.

§2. Establishment of the Difference Scheme

First we introduce some notations. Let $\Omega=[0,D]$, $Q_T=\Omega\times[0,T]$ be a rectangular region. We divide the domain Q_T into small grids by the parallel lines $x=x_j=jh$; $t=t_n=nk$ ($j=0,\cdots,J$; $n=0,\cdots,N$) where $Jh=D$, $Nk=T$. Let $Q_h=\{(x,t);\ x=jh,\ t=nk,\ j=0,\cdots,J;\ n=0,\cdots,N\}$, and let ϕ_j^n ($j=0,\cdots,J$; $n=0,\cdots,N$) denote the vector valued discrete function on the grid point (x_j,t_n).

Define

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$$\Delta_+V_j=V_{j+1}-V_j,\ \Delta_-V_j=V_j-V_{j-1},\ f_{\bar t}(x,t)=[f(x,t)-f(x,t-k)]/k.$$

We also introduce the inner product and norms appropriate to functions defined on the lattice Q_h. Suppose $f=(f_1,\cdots,f_M)^T$, $g=(g_1,\cdots,g_M)^T$. Then

$$(f,g)_h=\sum_{j=m}^{J-m}\langle f(x_j),g(x_j)\rangle h,$$

$$\|f\|_h^2=(f,f)_h,\quad \|f\|_\infty=\sup_{m<j<J-m}|f(x_j)|,$$

where $\langle f(x),g(x)\rangle=\sum_{l=1}^{M}f_l(x)\bar g_l(x)$, $|f|^2=\langle f,f\rangle$. The norms corresponding to the space of square integrable functions are

$$\|f\|^2_{L^2(\Omega)}=\int_0^D|f(x)|^2dx,\quad \|f\|_{L^\infty(\Omega)}=\operatorname*{esssup}_{x\in\Omega}|f(x)|.$$

Corresponding to (1.1)—(1.3), we construct following difference scheme

$$\begin{cases} i\dfrac{\phi_j^{n+1}-\phi_j^n}{k}+(-1)^m\dfrac{\Delta_+^m(A_{j-\frac{m}{2}}\Delta_-^m\phi_j^{n+\frac12})}{h^{2m}}+\beta_jP(|\phi_j^{n+1}|^2,|\phi_j^n|^2)\phi_j^{n+\frac12} \\ \quad +F_j^{n+\frac12}\cdot\phi_j^{n+\frac12}=0,\quad j=m,\cdots,J-m, & (2.1)\\ \phi_j^0=\phi_{hj},\quad j=0,\cdots,J, & (2.2)\\ \Delta_+^l\phi_0^n=\Delta_-^l\phi_J^n=0,\quad 0\le n\le N;\ l=0,\cdots,m-1, & (2.3)\end{cases}$$

where $\phi_j^{n+\frac12}=\frac12(\phi_j^{n+1}+\phi_j^n)$, $F_j^{n+\frac12}=F\left(x_j,t_n+\frac12k\right)$, $P(u,v)=(Q(u)-Q(v))/(u-v)$, $u\neq v$; $P(u,u)=q(u)$, $Q(z)=\int_0^z q(s)ds$, $\phi_{hj}=u_0(x_j)$ $(j=m,\cdots,J-m)$ and $\phi_{hj}=0$ $(j=0,\cdots,m-1;\ J-m+1,\cdots,J)$. For convenience, we will replace ϕ_j^n with ϕ_j^n, $u(x,t)$ with $u(x,t)$ and so on.

By the method in [7], we can get

Theorem 2.1. *The solution ϕ_j^n $(j=1,\cdots,J;\ n=1,\cdots,N)$ of the difference problem (2.1)—(2.3) exists, and is unique if $q(r)\in C^1[0,\infty)$ and k is small enough.*

§ 3. Priori Estimations for Difference Solution

In this section, we will get a series of priori estimates for the solution of difference equation (2.1)—(2.3).

Lemma 3.0. *For any $\{u_j\}$ and $\{v_j\}$ $(j=0,\cdots,J)$ there is a relation*

$$\sum_{j=0}^{J-1}u_j\Delta_+v_j=-\sum_{j=1}^{J}v_j\Delta_-u_j-u_0v_0+u_Jv_J.$$

Lemma 3.1. *Suppose $u_0(x)\in C(\Omega)$. Then there exists h_0, such that*

$$\|\phi_j^n\|_h^2=\|\phi_j^0\|_h^2\le 2\int_0^D|u_0(x)|^2dx=E_0,\quad n=0,\cdots,N;\ 0<h\le h_0. \tag{3.1}$$

The first equality of (3.1) indicates that the solution of the difference problem (2.1)—(2.3) is conservative of energy like the original problem.

Lemma 3.2. *Suppose the following conditions are satisfied*

(1) $\beta^*\ge\beta(x)\ge0$, *for* $x\in[0,D]$,

(2) *for any* $s\in[0,\infty)$, $q(s)\ge0$, $q'(s)\ge0$, $q(s)\in C^1$,

(3) *for any* $1\le l\le M$, $0\le x\le D$, $0<a_*\le a_l(x)\le a^*$,

$$\max_{1\le l,r\le M} 2|f_{l,r}(x,t)|\le F^*,\quad \max_{1\le l,r\le M} 2\left|\frac{\partial f_{l,r}}{\partial t}(x,t)\right|\le F^*,\quad (x,t)\in Q_T,$$

(4) $\dfrac{\partial^m u_0}{\partial x^m}\in C(\Omega)$.

Then we have the following estimate for the difference solution:

$$\left\|\frac{\Delta_+^m\phi^n}{h^m}\right\|_h\le E_1,\quad n=1,\cdots,N, \tag{3.2}$$

where E_1 is independent of h and k.

Proof. From the hypothesis of this lemma, we have $k_0>0$ such that

$$|f_{l,r,j}^{n+\frac12}|\le F^*,\ |(f_{l,r,j}^{n+\frac12})_{\bar t}|\le F^*,\quad n=1,\cdots,N-1;\ j=0,\cdots,J;\ l,r=1,\cdots,M;\ 0<k\le k_0,$$

and we can select $h_0>0$, such that

$$\sum_{j=m}^{J-m}\left|\frac{\Delta_+^m u_0(x_j)}{h^m}\right|^2 h\le 2\int_0^D\left|\frac{\partial^m u_0(x)}{\partial x^m}\right|^2 dx=K_2,\quad 0<h\le h_0.$$

Taking the inner product of $(\phi_j^{n+1})_{\bar t}$ with equation (2.1) yields

$$i\sum_{j=m}^{J-m}|(\phi_j^{n+1})_{\bar t}|^2+(-1)^m\sum_{j=m}^{J-m}\langle\Delta_+^m(A_{j-\frac m2}\Delta_-^m\phi_j^{n+\frac12}),(\phi_j^{n+1})_{\bar t}\rangle h/h^{2m}$$
$$+\sum_{j=m}^{J-m}\beta_j P(|\phi_j^{n+1}|^2,|\phi_j^n|^2)\langle\phi_j^{n+\frac12},(\phi_j^{n+1})_{\bar t}\rangle h+\sum_{j=m}^{J-m}\langle F_j^{n+\frac12}\phi_j^{n+\frac12},(\phi_j^{n+1})_{\bar t}\rangle h=0. \tag{3.3}$$

Consider every item in (3.3). Using Lemma 3.0 repeatedly and noticing condition (2.3), we have

$$(-1)^m\operatorname{Re}\sum_{j=m}^{J-m}\langle\Delta_+^m(A_{j-\frac m2}\Delta_-^m\phi_j^{n+\frac12}),(\phi_j^{n+1})_{\bar t}\rangle$$
$$=\sum_{j=m}^{J-m}\operatorname{Re}\langle A_{j+\frac m2}\Delta_+^m\phi_j^{n+\frac12},(\Delta_+^m\phi_j^{n+1})_{\bar t}\rangle=\frac12\sum_{j=m}^{J-m}\sum_{l=1}^M a_{l,j+\frac m2}|\Delta_+^m\phi_{l,j}^{n+1}|_{\bar t}^2. \tag{3.4}$$

Since

$$\operatorname{Re}[(\phi_{l,j}^{n+1}+\phi_{l,j}^n)(\bar\phi_{r,j}^{n+1})_{\bar t}]+\operatorname{Re}[(\phi_{r,j}^{n+1}+\phi_{r,j}^n)(\bar\phi_{l,j}^{n+1})_{\bar t}]=2[\operatorname{Re}(\phi_{l,j}^{n+1}\cdot\phi_{r,j}^{n+1})_{\bar t}],$$

from the symmetry of the matrix $F(x,t)$, we can get

$$\operatorname{Re}\Big[\sum_{j=m}^{J-m}\langle F_j^{n+\frac12}\phi_j^{n+\frac12},(\phi_j^{n+1})_{\bar t}\rangle\Big]=\frac12\sum_{j=m}^{J-m}\sum_{l=0}^M\sum_{r=0}^M[f_{l,r,j}^{n+\frac12}\operatorname{Re}(\phi_{l,j}^{n+1}\bar\phi_{r,j}^{n+1})]_{\bar t}$$
$$-\frac12\sum_{j=m}^{J-m}\sum_{l=1}^M\sum_{r=1}^M(f_{l,r,j}^{n+\frac12})_{\bar t}\cdot\operatorname{Re}(\phi_{l,j}^n\cdot\bar\phi_{r,j}^n), \tag{3.5}$$

while

$$P(|\phi_j^{n+1}|^2,|\phi_j^n|^2)\cdot\operatorname{Re}\langle\phi_j^{n+\frac12},(\phi_j^{n+1})_{\bar t}\rangle=Q(|\phi_j^{n+1}|^2)_{\bar t}. \tag{3.6}$$

Substituting (3.4)—(3.6) into (3.3) and taking the real part, we have

$$\sum_{j=m}^{J-m}\sum_{l=1}^M a_{l,j+\frac m2}\left|\frac{\Delta_+^m\phi_{l,j}^{n+1}}{h^m}\right|_{\bar t}^2+\sum_{j=m}^{J-m}\beta_j[Q(|\phi_j^{n+1}|^2)]_{\bar t}+\sum_{j=m}^{J-m}\sum_{l,r=1}^M(f_{l,r,j}^{n+\frac12}\operatorname{Re}\phi_{l,j}^{n+1}\cdot\bar\phi_{r,j}^{n+1})_{\bar t}$$
$$=\sum_{j=m}^{J-m}\sum_{l,r=1}^M(f_{l,r,j}^{n+\frac12})_{\bar t}\operatorname{Re}(\phi_{l,j}^n\cdot\bar\phi_{r,j}^n).$$

Summing up for $n=0,\cdots,N_0-1$, we get

$$\sum_{j=m}^{J-m}\sum_{l=1}^M a_{l,j+\frac m2}\left|\frac{\Delta_+^m\phi_{l,j}^{N_0}}{h^m}\right|^2 h+\sum_{j=m}^{J-m}\beta_j Q(|\phi_j^{N_0}|^2)h+\sum_{j=m}^{J-m}\sum_{r,l=1}^M f_{l,r,j}^{N_0-\frac12}\operatorname{Re}(\phi_{l,j}^{N_0}\cdot\bar\phi_{r,j}^{N_0})h$$
$$=\sum_{j=m}^{J-m}\sum_{l=1}^M a_{l,j+\frac m2}\left|\frac{\Delta_+^m\phi_{l,j}^0}{h^m}\right|^2 h+\sum_{j=m}^{J-m}\beta_j Q(|\phi_j^0|)h$$

$$+\sum_{j=m}^{J-m}\sum_{l,r=1}^{M} f_{l,r,j}^{-\frac{1}{2}}\operatorname{Re}(\phi_{l,j}^{0}\phi_{r,j}^{0})-\sum_{n=0}^{N_0-1}\sum_{j=m}^{J-m}\sum_{l,r=1}^{M}(f_{l,r,j}^{n+\frac{1}{2}})_{\bar{t}}\operatorname{Re}(\phi_{l,j}^{n}\bar{\phi}_{r,j}^{n}). \tag{3.7}$$

It follows that

$$\sum_{j=m}^{J-m}\sum_{l=1}^{M}\left|\frac{\Delta_+^m\phi_{l,j}^{N_0}}{h^m}\right|^2 h+\sum_{j=m}^{J-m}\beta_j Q(|\phi_j^{N_0}|^2)h$$
$$\leqslant (a^*K_2+\beta^*Q_0+2F^*ME_0+F^*MTE_0)/a_*\triangleq E_1,\quad Q_0\triangleq Q(\max_{x\in\Omega}|u_0(x)|^2).$$

The proof is completed.

Making use of Sobolev's inequality[8], we can obtain

Lemma 3.3. *Suppose that the conditions for Lemma 3.2 are satisfied. Then there is a constant E_2 independent of h and k, such that*

$$\sum_{l=1}^{m-1}\left\|\frac{\Delta_+^l\phi^n}{h^l}\right\|_\infty\leqslant E_2,\quad \|q(|\phi^n|^2)\|_\infty\leqslant E_2,\quad n=0,\cdots,N.$$

Next, we will continue discussing the priori estimation of the solution of the difference equation. Let

$$z_j^{n+1}=(\phi_j^{n+1}-\phi_j^n)/k.$$

From equations (2.1) and (2.3), we have

$$i\frac{z_j^{n+1}-z_j^n}{k}+(-1)^m\frac{\Delta_+^m(A_{j-\frac{m}{2}}\Delta_-^m z_j^{n+\frac{1}{2}})}{h^{2m}}+\beta_j P(|\phi_j^{n+1}|^2,\ |\phi_j^n|^2)\phi_j^{n+\frac{1}{2}}$$
$$-\beta_j P(|\phi_j^{n-1}|^2,\ |\phi_j^n|^2)\phi_j^{n-\frac{1}{2}}+(F_j^{n+\frac{1}{2}})_{\bar{t}}\phi_j^{n+\frac{1}{2}}+F_j^{n+\frac{1}{2}}z_j^{n+\frac{1}{2}}=0,$$
$$n=1,\cdots,N-1;\ j=m,\cdots,J-m, \tag{3.8}$$
$$\Delta_+^l z_0^{n+1}=\Delta_-^l z_J^{n+1}=0,\quad n=0,\cdots,N-1. \tag{3.9}$$

On the other hand, we define

$$\begin{cases} z_j^0=(-1)^m i\dfrac{\Delta_+^m(A_{j-\frac{m}{2}}\Delta_-^m\phi_j^0)}{h^{2m}}+i\beta_j q(|\phi_j^0|^2)\phi_j^0+iF_j^{-\frac{1}{2}}\cdot\phi_j^0,\quad j=m,\cdots,J-m; & (3.10)\\ z_j^0=0,\ j=0,\cdots,m-1;\ J-m+1,\cdots,J. \end{cases}$$

According to (2.1), when $n=0$ and (3.10), we have

$$\begin{cases} i\dfrac{z_j^1-z_j^0}{k}+(-1)^m\dfrac{\Delta_+^m(A_{j-\frac{m}{2}}\Delta_-^m z_j^1)}{h^{2m}}+\beta_j[P(|\phi_j^1|^2,\ |\phi_j^0|^2)-q(|\phi_j^0|^2)]\phi_j^{\frac{1}{2}} \\ \quad+\dfrac{\beta_j}{2}q(|\phi_j^0|^2)z_j^1+(F_j^{\frac{1}{2}})_{\bar{t}}\phi_j^0+\dfrac{1}{2}F_j^{\frac{1}{2}}z_j^1=0,\quad j=m,\cdots,J-m, & (3.11)\\ \Delta_+^l z_0^1=\Delta_-^l z_J^1=0,\quad l=0,\cdots,m-1. \end{cases}$$

Therefore, we obtain the system of equations (3.8)—(3.11) about $\{z_j^n\}$ $(j=0,\cdots,J;\ n=0,\cdots,N)$.

Lemma 3.4. *Suppose that the conditions for Lemma 3.2 hold, and assume*

(1) $\dfrac{\partial^m}{\partial x^m}\left(A(x)\dfrac{\partial^m u_0}{\partial x^m}\right)\in C(\Omega)$, $q(|u_0|^2)\in C(Q_T)$, $K_4=\max\limits_{0<\xi\leqslant E_2^2}|q'(\xi)|$,

(2) $\dfrac{\partial f_{l,r}}{\partial t}(x,t)\in C(Q_T)$, $l,r=1,\cdots,M$; $K_5=2\max\limits_{(x,t)\in Q_T}\left\|\dfrac{\partial F(x,t)}{\partial t}\right\|_2$.

Then there exists a constant E_3 independent of h and k, such that

$$\|z_j^{n+1}\|_h^2=\sum_{j=m}^{J-m}\left|\frac{\phi_j^{n+1}-\phi_j^n}{k}\right|^2 h\leqslant E_3,\quad n=0,\cdots,N, \tag{3.12}$$

if $1-2k(8E_0^2K_4\beta^*+K_5)>0$.

Proof. From condition (1) and equation (3.10), there exists a constant K_8 independent of h and k, such that

$$\|z^0\|_h\leqslant K_8.$$

Mean while, according to condition (2), we can find a constant $k_0>0$, such that

$$\|(F_j^{n+\frac{1}{2}})_l\|_2\leqslant K_5,\quad n=0,\cdots,N-1;\ 0<k\leqslant k_0.$$

(a) Making the inner product of z_j^1 with equation (3.11) and taking the imaginary part, we have

$$\sum_{j=m}^{J-m}|z_j^1|^2h\leqslant\sum_{j=m}^{J-m}|z_j^0|^2h+2k\beta^*\sum_{j=m}^{J-m}h\,|P(|\phi_j^1|^2,\ |\phi_j^0|^2)-q(|\phi_j^0|^2)|\,|\langle\phi_j^0,\ z_j^1\rangle|$$
$$+2\sum_{j=m}^{J-m}|\langle(F_j^{\frac{1}{2}})_l\phi_j^0,\ z_j^1\rangle|h. \tag{3.13}$$

Since

$$|P(|\phi_j^1|^2,\ |\phi_j^0|^2)-q(|\phi_j^0|^2)|$$
$$=|q'(\xi_j)|\,(|\phi_j^1|^2-|\phi_j^0|^2)\leqslant 2K_4E_0|z_j^1|,\quad \xi_j\in(|\phi_j^0|^2,\ |\phi_j^1|^2), \tag{3.14}$$

$$|\langle(F_j^{\frac{1}{2}})_l\phi_j^0,\ z_j^1\rangle|\leqslant\|(F_j^{\frac{1}{2}})_l\|_2\cdot|\phi_j^0|\cdot|z_j^1|\leqslant\frac{1}{2}K_5(|\phi_j^0|^2+|z_j^1|^2), \tag{3.15}$$

substituting (3.14) and (3.15) into (3.13), we obtain

$$[1-k(8E_0^2K_4\beta^*+K_5)]\sum_{j=m}^{J-m}h|z_j^1|^2\leqslant K_8+\frac{1}{2}K_8E_0k.$$

It follows that there is a constant K_6, such that

$$\|z_j^1\|_h^2\leqslant K_6.$$

(b) Making the inner product of $z_j^{n+\frac{1}{2}}$ with equations (3.8) and taking the imaginary part, there is

$$\frac{1}{2}h\sum_{j=m}^{J-m}|z_j^{n+1}|_i^2\leqslant 4\beta^*E_2^2K_4\sum_{j=m}^{J-m}(|z_j^{n+1}|^2+|z_j^n|^2)h$$
$$+\frac{1}{2}K_5E_0+K_5\sum_{j=m}^{J-m}(|z_j^{n+1}|^2+|z_j^n|^2)h.$$

Summing up for $n=1,\cdots,N_0-1$, we have

$$\sum_{j=m}^{J-m}|z_j^{N_0}|^2h\leqslant K_6+K_5E_0+2k(8E_0^2K_4\beta^*+K_5)\sum_{n=1}^{N_0}\sum_{j=m}^{J-m}|z_j^n|^2h.$$

By Bellman's inequality in discrete form, we can get

$$\sum_{j=m}^{J-m}|z_j^{N_0}|^2h\leqslant(K_5E_0+K_6)\exp[(16E_0^2K_4\beta^*+2K_5)T],$$

which implies (3.12).

Lemma 3.5. *Under the conditions for Lemma* 3.4, *and assuming*

$$\frac{\partial^m a_l(x)}{\partial x^m}\in C(\Omega),\quad l=1,\cdots,M,$$

there exists a constant $E_4>0$ *independent of* h *and* k, *such that*

$$\sum_{j=m}^{J-m}\left|\frac{\Delta_+^l(A_{j-\frac{m}{2}}\Delta^m\phi_j^{n+\frac{1}{2}})}{h^{m+l}}\right|^2h\leqslant E_4,\quad n=0,\cdots,N-1;\ l=1,\cdots,M. \tag{3.16}$$

Proof. The estimate (3.16) follows immediately from the system (2.1) and Sobolev's inequality.

Finally, as a side product of the above estimations, we can show the existence and uniqueness of the generalized solution of problem (1.1)—(1.3) if we make use of the method in [7]. The generalized solution

$$u(x, t)\in Z=L^\infty(0, T; W^{2m,2}(\Omega))\cap L^\infty(0, T; W^{2m-1,\infty}(\Omega))\cap W^{1,\infty}(0, T; L^2(\Omega)),$$

which depends on the initial data.

§ 4. Convergence and Stability

Theorem 4.1. *Suppose that $u(x, t)$ is the classical solution of problem* (1.1)—(1.3) *and suppose*

(1) $u(\cdot, t)\in C^{2(m+1)}[0, D]$, $\forall t\in[0, T]$; $u(x, \cdot)\in C^3[0, T]$, $\forall x\in[0, D]$,

(2) $a_l(x)\in C^{m+2}[0, D]$, $l=1, \cdots, M$,

(3) *the conditions for Lemma* 3.2 *hold.*

Then the difference solution of the problem (2.1)—(2.3) *converges to the classical smooth solution of problem* (1.1)—(1.3) *in the sense of* $\|\cdot\|_h$.

Proof. Taking $K_8=\max\limits_{(x,t)\in Q_T}|u(x, t)|$, $K_9=\max\limits_{0<\xi<\max(E_1^2, K_8^2)}|q'(\xi)|$. Let $u_j^n=u(x_j, t_n)$ and substitute it into the difference equation (2.1). By Taylor's expansion, we have

$$i\frac{u_j^{n+1}-u_j^n}{k}+(-1)^m\frac{\Delta_+^m(A_{j-\frac{m}{2}}\Delta_-^m u_j^{n+\frac{1}{2}})}{h^{2m}}+\beta_j P(|u_j^{n+1}|^2, |u_j^n|^2)u_j^{n+\frac{1}{2}}=O(h^2+k^2).$$

Let $e_j^n=u_j^n-\phi_j^n$. Then e_j^n satisfies

$$\begin{cases} i(e_j^{n+1})_{\bar t}+(-1)^m\dfrac{\Delta_+(A_{j-\frac{m}{2}}\Delta_-^m e_j^{n+\frac{1}{2}})}{h^{2m}}+F_j^{n+\frac{1}{2}}e_j^{n+\frac{1}{2}}+\beta_j[P(|u_j^{n+1}|^2, |u_j^n|^2)u_j^{n+\frac{1}{2}} \\ \quad -P(|\phi_j^{n+1}|^2, |\phi_j^n|^2)\phi_j^{n+\frac{1}{2}}]=O(h^2+k^2),\quad j=m, \cdots, J-m, & (4.1) \\ e_j^0=u(x_j)-\phi_{hj}=O(h),\quad j=0, \cdots, J, & (4.2) \\ \Delta_+^l e_0^n=O(h^{l+1}),\ \Delta_-^l e_J^n=O(h^{l+1}),\quad n=0, \cdots, N;\ l=0, \cdots, m-1. & (4.3) \end{cases}$$

Making the inner product of $e_j^{n+\frac{1}{2}}$ with (4.1) and taking the imaginary part gives

$$\begin{aligned} &\sum_{j=m}^{J-m}h|e_j^{n+1}|_{\bar t}^2+(-1)^m\sum_{j=m}^{J-m}h\cdot\mathrm{Im}\left\langle\frac{\Delta_+^m(A_{j-\frac{m}{2}}\Delta_-^m e_j^{n+\frac{1}{2}})}{h^{2m}}, e_j^{n+\frac{1}{2}}\right\rangle \\ &\qquad+\sum_{j=m}^{J-m}\beta_j h[P(|u_j^{n+1}|^2, |u_j^n|^2)-P(|\phi_j^{n+1}|^2, |\phi_j^n|^2)]\cdot\mathrm{Im}\langle\phi_j^{n+\frac{1}{2}}, e_j^{n+\frac{1}{2}}\rangle \\ &=\sum_{j=m}^{J-m}\mathrm{Im}\,\langle O(h^2+k^2), e_j^{n+\frac{1}{2}}\rangle h. \end{aligned}\tag{4.4}$$

By the hypothesis of the theorem and boundary condition (4.3), we have

$$\sum_{j=m}^{J-m}\left\langle\frac{\Delta_+^m(A_{j-\frac{m}{2}}\Delta_-^m e_j^{n+\frac{1}{2}})}{h^{2m}}, e_j^{n+\frac{1}{2}}\right\rangle=(-1)^m\sum_{j=m}^{J-m}\left\langle\frac{A_{j+\frac{m}{2}}\Delta_+^m e_j^{n+\frac{1}{2}}}{h^m}, \frac{\Delta_+^m e_j^{n+\frac{1}{2}}}{h^m}\right\rangle+O(h),\tag{4.5}$$

$$\begin{aligned}
&\left|[P(|u_j^{n+1}|^2, |u_j^n|^2)-P(|\phi_j^{n+1}|^2, |\phi_j^n|^2)]\langle\phi_j^{n+\frac{1}{2}}, e_j^{n+\frac{1}{2}}\rangle\right| \\
&=\left|[P(|u_j^{n+1}|^2, |u_j^n|^2)-P(|u_j^n|^2, |\phi_j^{n+1}|^2)+P(|u_j^n|^2, |\phi_j^{n+1}|^2)\right. \\
&\quad \left.-P(|\phi_j^{n+1}|^2, |\phi_j^n|^2)]\langle\phi_j^{n+\frac{1}{2}}, e_j^{n+\frac{1}{2}}\rangle\right| \\
&=\left|q'(\xi_j^n)(|u_j^{n+1}|^2-|\phi_j^{n+1}|^2)+q'(\eta_j^n)(|u_j^n|^2-|\phi_j^n|^2)\right|\cdot\left|\langle\phi_j^{n+\frac{1}{2}}, e_j^{n+\frac{1}{2}}\rangle\right| \\
&\leqslant 2K_9(|e_j^{n+1}|+|e_j^n|)(K_8+E_2)E_2|e_j^{n+\frac{1}{2}}| \\
&\leqslant 2K_9E_2(K_8+E_2)(|e_j^{n+1}|^2+|e_j^n|^2),
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
|\langle O(h^2+k^2), e_j^{n+\frac{1}{2}}\rangle| &\leqslant \frac{1}{2}[(O(h^2+k^2))^2+|e_j^{n+\frac{1}{2}}|^2] \\
&\leqslant \frac{1}{2}(|e_j^{n+1}|^2+|e_j^n|^2)+O((h^2+k^2)^2),
\end{aligned} \tag{4.7}$$

where $\xi_j^{n+1}\in(|u_j^{n+1}|^2, |\phi_j^{n+1}|^2)$, $\eta_j^n\in(|u_j^n|^2, |\phi_j^n|^2)$, $j=m, \cdots, J-m$.

Substituting (4.5)—(4.7) into (4.4), we obtain

$$\sum_{j=m}^{J-m}|e_j^{n+1}|_{\bar{t}}^2h\leqslant O(h)h+(2\beta^*K_9E_2(K_8+E_2)+1)\sum_{j=m}^{J-m}(|e_j^n|^2+|e_j^{n+1}|^2)h+O((h^2+k^2)^2).$$

Summing up for $n=0, \cdots, N_0-1$, there is

$$\begin{aligned}
\sum_{j=m}^{J-m}|e_j^{N_0}|^2h\leqslant&\sum_{j=m}^{J-m}|e_j^0|^2h+T\cdot O((h^2+k^2)^2)+T\cdot O(h^2) \\
&+2k[2\beta^*K_9E_2(K_8+E_2)+1]\sum_{n=0}^{N_0}\|e^n\|_h^2.
\end{aligned}$$

By Bellman's inequality, it is easy to see that if $N_0k\leqslant T$ and $1-2k(2\beta^*K_9E_2(K_8+E_2)+1)>0$, then there is a constant K_{10} independent of h and k, such that

$$\sum_{j=m}^{J-m}|e_j^{N_0}|^2h\leqslant K_{10}\Big(\sum_{j=m}^{J-m}|e_j^0|^2h+O((h^2+k^2)^2)+O(h^2)\Big).$$

This inequality indicates that the conclusion of the theorem is true.

Theorem 4.2. *Under the conditions for Lemma 3.2, the difference solution of the problem* (2.1)—(2.3) *is stable in the sense of square norm.*

Proof. Suppose both ϕ_j^n and ψ_j^n satisfy equation (2.1) and boundary condition (2.2) and

$$\phi_j^0=\phi_{hj},\ \psi_j^0=\psi_{hj},\quad j=0, \cdots, J.$$

Let $\varepsilon_j^n=\phi_j^n-\psi_j^n$, then ε_j^n satisfies the following equations

$$\begin{cases}
i(\varepsilon_j^{n+1})_{\bar{t}}+(-1)^m\dfrac{\Delta_+^m(A_{j-\frac{m}{2}}\Delta_-^m\varepsilon_j^{n+\frac{1}{2}})}{h^{2m}}+F_j^{n+\frac{1}{2}}\varepsilon_j^{n+\frac{1}{2}}+\beta_j[P(|\phi_j^{n+1}|^2, |\phi_j^n|^2)\phi_j^{n+\frac{1}{2}} \\
\quad -P(|\psi_j^{n+1}|^2, |\psi_j^n|^2)\psi_j^{n+\frac{1}{2}}]=0,\quad j=m, \cdots, J-m;\ n=0, \cdots, N-1, \\
\varepsilon_j^0=\phi_{hj}-\psi_{hj},\quad j=0, \cdots, J, \\
\Delta_+^l\varepsilon_0^n=\Delta_-^l\varepsilon_J^n=0,\quad l=0, \cdots, m-1;\ n=0, \cdots, N.
\end{cases}$$

The rest of the proof is similar to that of Theorem 4.1.

Using similar methods, we can also prove that all the results of Sections 2—4 hold for the periodic boundary and initial value problem of equations (1.1).

§ 5. Numerical Examples

(1) Consider the following problem

$$\begin{cases} iu_t + u_{xxxx} + 6|u|^2 u - 150(\sin^2 x)u = 0, \\ u(x,\ 0) = \left(\frac{1}{2}5\sqrt{2}\right)(1+i)\sin x, \\ u(0,\ t) = u(\pi,\ t) = 0, \\ u_{xx}(0,\ t) = u_{xx}(\pi,\ t) = 0. \end{cases}$$

It has a classical solution

$$u = u(x,\ t) = 5\exp\left(i\left(t + \frac{1}{4}\pi\right)\right)\sin x.$$

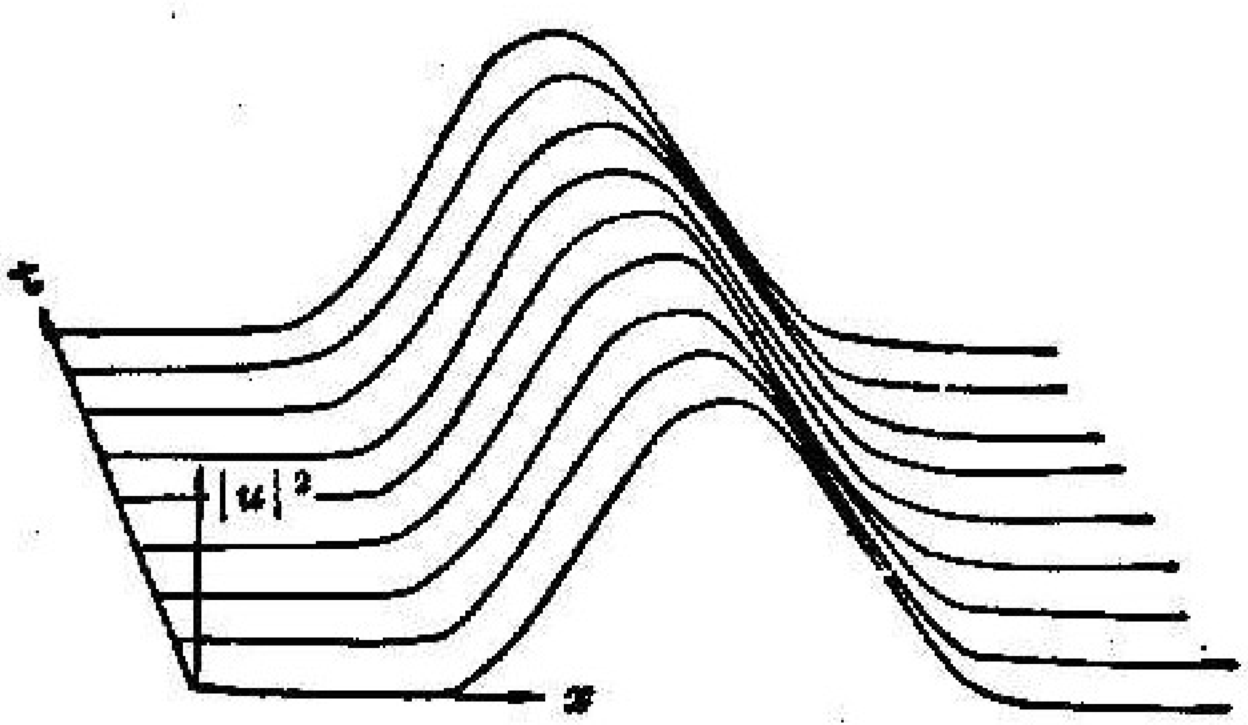

Fig. 1

The numerical results are written in Table 1 and the pattern of the solution is described in Fig. 1.

Table 1 Result at $t=1$, $x=$ /100

	Classical solu. $\|u\|^2$	Num. solu. $\|u_h^1\|^2$, $\Delta t=10^{-5}$	Num. solu. $\|u_h^2\|^2$, $\Delta t=10^{-4}$
$\pi/10$	2.387287571	2.3872876	2.3872871
$2\pi/10$	8.637287571	8.6372881	8.6372847
$3\pi/10$	1.636271243×10	1.6362712×10	1.6362710×10
$4\pi/10$	2.261271243×10	2.2612712×10	2.2612712×10
$5\pi/10$	2.5×10	2.5×10	2.5000001×10
$6\pi/10$	2.261271243×10	2.2612713×10	2.2612711×10
$7\pi/10$	1.636271243×10	1.6362712×10	1.6362710×10
$8\pi/10$	8.637287571	8.6372880	8.6372857
$9\pi/10$	2.387287571	2.3872876	2.3872878
π	0	0	0

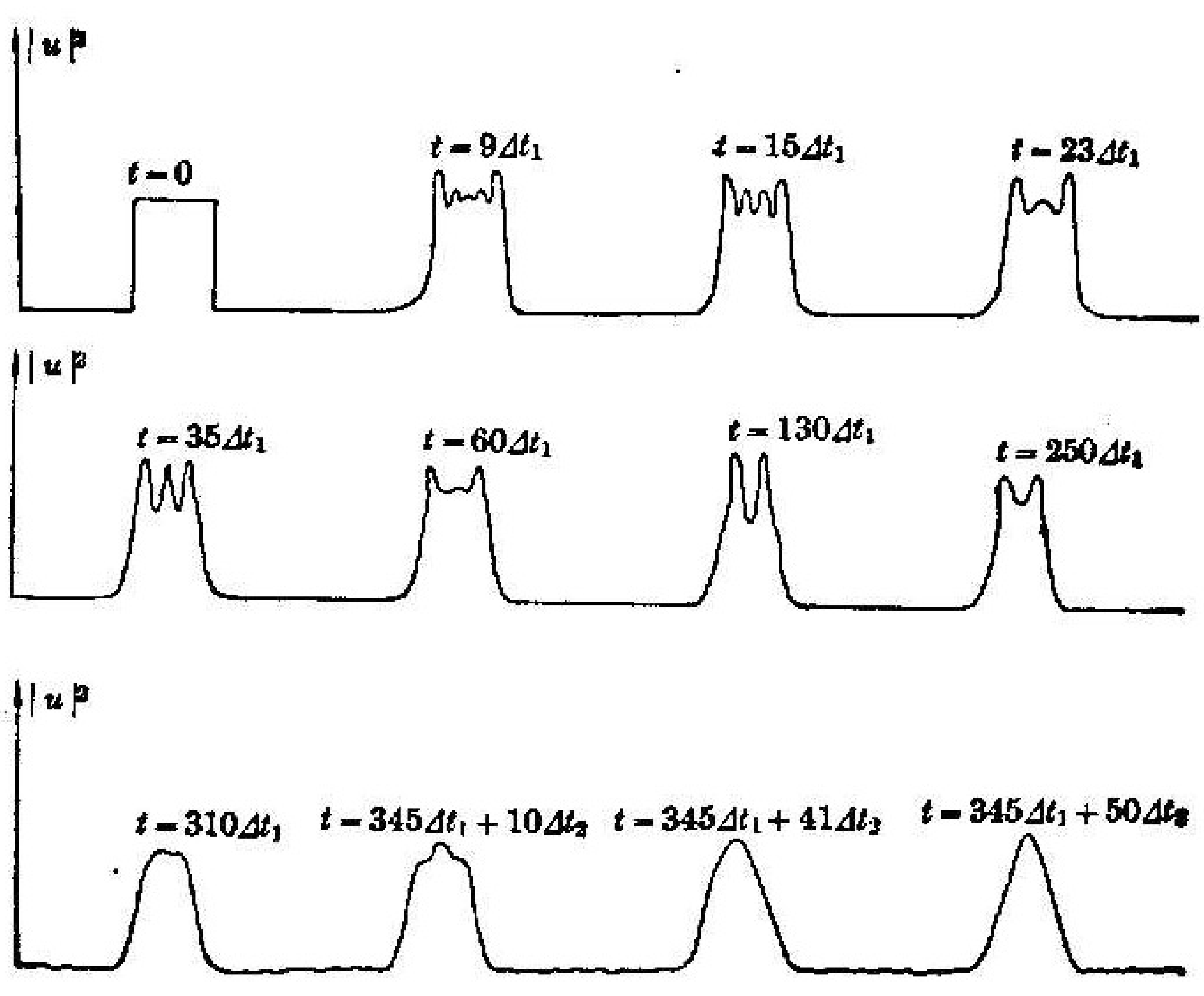

Fig. 2 $\Delta t_1 = 0.4\times 10^{-6}$, $\Delta t_2 = 2\Delta t_1$

(2) The second example is

$$\begin{cases} iu_t+u_{xxxx}+6|u|^2u=0, \\ u(x, 0)=\begin{cases} 3.5+5i, & x\in[1.48, 2.52]; \\ 0, & x\in[0, 1.48)\cup(2.52, 4], \end{cases} \\ u(x+4, t)=u(x, t). \end{cases}$$

The numerical results are depicted in Fig. 2.

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