# NONCONFORMING FINITE ELEMENTS WITH COMPENSATION FOR PLATE BENDING PROBLEMS*

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## § 1. Introduction

As many authors pointed out, to solve plate bending problems and other elliptic boundary value problems of equations with order higher than two it is much more convenient to use nonconforming finite elements, for they have less degrees of freedom, relatively simpler base functions and better convergence behavior when convergence occurs. There are two methods, among others, which use the nonconforming elements. One is the so called tolerance method[3], in which we calculate the bilinear form of the variational problem element by element and then take the sum (briefly, elements sum) as the value of the bilinear function on the whole domain, just as what we do in using conforming elements. With this method, the convergence of the finite element approximations should be analysed carefully, since they are not always convergent to the solution of the original problem as the mesh gets finer. For the method many elements have been analysed, see, for example, [2, 5, 7] and [9]. But all these analyses show that the convergence order is lower than that of the conforming elements with the same degree of piecewise polynomial interpolation. This is because the elements sum is used to substitute the real bilinear form. The other method is the penalty method. As shown in [1, 3] and [8], the convergence always occurs, but its order is only half of that of the conforming elements with the same degree of piecewise polynomials.

To improve the accuracy order, we will give a compensation method in this paper. The main idea of the method is to add something to the elements sum so that the error caused by the substitution of this sum for the bilinear function can be compensated. This method gives better approximations than both the tolerance method and the penalty method under certain conditions. Moreover, if the elements used are the so called weakly discontinuous elements[3], the method gives approximations of the same accuracy order with that of the conforming elements. In addition it has the advantage that no Lagrange multipliers or other additional parameters are used, so no additional degrees of freedom will be introduced. Hence the amount of computations will not be increased.

The paper is outlined as follows: In § 2, a variational model with compensation of the clamped plate bending problem will be described, and the existence and uniqueness of the solution of the proposed variational problem are analysed. In § 3, the error estimates are given and the theorem of convergence is proved. In § 4, we

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will deal with plates with other kinds of boundary conditions. Finally, some examples are given in § 5.

## § 2. The Compensation Method for Clamped Plates

Let us consider an elastic thin plate of a convex polygonal domain $\Omega$ in $R^2$. A mathematical model of the bending problem of the plate with clamped boundary is

$$\begin{cases} \Delta^2 u = f, & (x_1, x_2) \in \Omega, \quad (2.1) \\ u = \dfrac{\partial u}{\partial n} = 0, & (x_1, x_2) \in \partial\Omega, \quad (2.2) \end{cases}$$

where $n$ is the outward normal direction of boundary $\partial\Omega$. Set

$$a(u, v) = \int_\Omega (u_{x_1x_1} v_{x_1x_1} + 2u_{x_1x_2} v_{x_1x_2} + u_{x_2x_2} v_{x_2x_2})\, dx,$$

where $dx = dx_1 dx_2$, $u_{x_1x_2} = \dfrac{\partial^2 u}{\partial x_1 \partial x_2}$, etc. We can associate with (2.1)—(2.2) a variational problem:

$$\text{Find } u \in H_0^2(\Omega), \text{ such that } a(u, v) = (f, v)_0 = \int_\Omega f v\, dx, \quad \forall v \in H_0^2(\Omega). \quad (2.3)$$

As is well known, $a(u, v)$ is $H_0^2(\Omega)$-elliptic, i.e. there exists a constant $m>0$ independent of $u$, such that

$$m\|u\|_{2,\Omega}^2 \leqslant a(u, u), \quad \forall u \in H_0^2(\Omega). \quad (2.4)$$

Also, $a(u, v)$ is bounded in $H^2(\Omega)$, i.e. there exists a constant $M$ independent of $u$ and $v$, such that

$$|a(u, v)| \leqslant M\|u\|_{2,\Omega}\|v\|_{2,\Omega}.$$

Hence problem (2.3) has a unique solution $u$, which is called the weak solution of (2.1)—(2.2).

We are going to consider solving (2.3) approximately by the finite element method. As usual, put a triangulation on $\Omega$ and let $\Omega_h$ be the set of all triangles (elements) obtained. For $\sigma \in \Omega_h$, set

$h_\sigma$ = the diameter of $\sigma$,

$\rho_\sigma$ = the diameter of the inscribed circle in $\sigma$.

Let

$$h = \max_{\sigma \in \Omega_h} h_\sigma, \quad \rho = \min_{\sigma \in \Omega_h} \rho_\sigma.$$

Next, we will consider a family of triangulations on $\Omega$, which will be called a regular family if each of its triangulation satisfies

$$\frac{h}{\rho} \leqslant C, \quad (2.5)$$

a constant being independent of the triangulation. By the way, letters $c$ and $C$ will be used as generic constants which may take different values at different places. We will always assume that $h \leqslant 1$. Now, construct a finite element space (i.e. a piecewise polynomial space) $V_h$ associated with the decomposition of $\Omega$. Let $V_h(\sigma)$ be the set of all restrictions on $\sigma$ of functions in $V_h$ and $P_k(\sigma)$ the set of all polynomials of degree $\leqslant k$ on $\sigma$. Suppose $P_k(\sigma) \subset V_h(\sigma)$. In general, we do not make the assumption $V_h \subset H_0^2(\Omega)$, i.e. $V_h$ is a nonconforming finite element space. For $v \in V_h$, set

$$|v|_{l,\sigma}^2=\sum_{|\alpha|=l}\int_\sigma (D^\alpha v)^2\,dx,$$

$$|v|_{l,h}^2=\sum_{\sigma\in\Omega_h}|v|_{l,\sigma}^2,$$

$$\|v\|_{m,h}^2=\sum_{l=0}^{m}|v|_{l,h}^2.$$

Let $T_h$ be the set of interelement sides or, sometimes, the union of these sides. Corresponding to the common side of two adjacent elements $v\in V_h$ has two traces $v^+$ and $v^-$ on $\tau$. Set

$$[v]=v^+-v^-. \tag{2.6}$$

It is clear that $[v]$ is a function defined on $T_h$. Suppose

$$a_h(u,v)=\sum_{\sigma\in\Omega_h}\int_\sigma (u_{x_1x_1}v_{x_1x_1}+2u_{x_1x_2}v_{x_1x_2}+u_{x_2x_2}v_{x_2x_2})\,dx.$$

In the tolerance method, or if $V_h$ is a conforming finite element space, the approximation $u_h$ of the solution $u$ of (2.3) is obtained by solving the problem:

$$\text{Find } u_h\in V_h \text{ such that } a_h(u_h,v)=(f,v)_0,\quad \forall v\in V_h. \tag{2.7}$$

Assume $V_h\subset C(\Omega)$ and $v|_{\partial\Omega}=0$, $\forall v\in V_h$. Define a bilinear form

$$c(u,v)=\sum_{\tau\in T_h}\int_\tau\left(\tilde{u}_{nn}[v_n]+\tilde{v}_{nn}[u_n]+\frac{\gamma}{h}[u_n][v_n]\right)ds,$$

where $u_n=\dfrac{\partial u}{\partial n}$, $u_{nn}=\dfrac{\partial^2 u}{\partial n^2}$ and

$$\tilde{u}_{nn}=\frac{1}{2}(u_{nn}^+ + u_{nn}^-),$$

and $n$ is the normal direction of $\tau$ pointing to the "+" hand. $\gamma>0$ is a constant, independent of $h$, to be determined later. Set

$$b(u,v)=a_h(u,v)+c(u,v)-\int_{\partial\Omega}\left(u_{nn}v_n+v_{nn}u_n+\frac{\gamma_1}{h}u_nv_n\right)ds.$$

Instead of (2.7), problem

$$\text{Find } u_h\in V_h, \text{ such that } b(u_h,v)=(f,v)_0,\quad \forall v\in V_h \tag{2.8}$$

will be used to get our finite element approximation $u_h$. Clearly, (2.8) becomes (2.7) when $V_h\subset H_0^2(\Omega)$.

To study the properties of the bilinear form $b(u,v)$, we need the following lemma.

**Lemma 2.1.** *Assume $v\in V_h$, $\sigma\in\Omega_h$. Then there exists a constant $C$ independent of $v$, $\sigma$ and $h$, such that*

$$\int_{\partial\Omega} v^2 ds\leqslant Ch^{-1}\int_\sigma v^2\,dx. \tag{2.9}$$

The lemma can be proved easily by transforming $\sigma$ in to the unit triangle and using the trace theorem and inverse inequalities.

We assume that $a_h(u,v)$ is uniformly $V_h$-elliptic, i.e. there exists a constant $\alpha>0$ independent of $v$ and $h$, such that

$$\alpha\|v\|_{2,h}^2\leqslant a_h(v,v),\quad \forall v\in V_h. \tag{2.10}$$

Notice, for $u\in V_h$,

$$c(u, u)=\sum_{\tau\in T_h}\int_\tau\left\{(u_{nn}^+-u_{nn}^-)[u_n]+\frac{\gamma}{h}[u_n]^2\right\}ds.$$

If $\varepsilon$ is an arbitrary positive real number, we have

$$\int_\tau |u_{nn}^+[u_n]|ds\leqslant\frac{\varepsilon}{2}\int_\tau (u_{nn}^+)^2ds+\frac{1}{2\varepsilon}\int_\tau [u_n]^2ds.$$

Using Lemma 2.1, we then have

$$\int_\tau |u_{nn}^+[u_n]|ds\leqslant\frac{C\varepsilon}{2h}|u|_{2,\sigma_1}^2+\frac{1}{2\varepsilon}\int_\tau [u_n]^2ds,$$

where $\sigma_1$ and $\sigma_2$ are the two elements lying at the "+" and "−" sides of $\tau$ respectively. Let

$$|v|_{l,\delta\tau}^2=|v|_{l,\sigma_1}^2+|v|_{l,\sigma_2}^2.$$

Then

$$\int_\tau |(u_{nn}^++u_{nn}^-)[u_n]|ds\leqslant\frac{C\varepsilon}{2h}|u|_{2,\delta\tau}^2+\frac{1}{\varepsilon}\int_\tau [u_n]^2ds.$$

Thus

$$\sum_{\tau\in T_h}\int_\tau |(u_{nn}^++u_{nn}^-)[u_n]|ds\leqslant\frac{C_1\varepsilon}{h}|u|_{2,h}^2+\frac{1}{\varepsilon}\sum_{\tau\in T_h}\int_\tau [u_n]^2ds,$$

where $C_1=\frac{3}{2}C$, since each seminorm $|u|_{2,\sigma}^2$ is repeated at most 3 times when we take the sum. If we take $\varepsilon=\frac{\alpha h}{3C_1}$, where $\alpha$ is the constant in (2.10), then $C_1\varepsilon/h=\alpha/3$. Choose $\gamma\geqslant 3C_1/\alpha$, i.e. $\frac{\gamma}{h}\geqslant\frac{3C_1}{\alpha h}=\frac{1}{\varepsilon}$. Thus we see

$$c(u, u)\geqslant-\frac{\alpha}{3}|u|_{2,h}^2+\left(\frac{\gamma}{h}-\frac{1}{\varepsilon}\right)\sum_{\tau\in T_h}\int_\tau [u_n]^2ds\geqslant-\frac{\alpha}{3}|u|_{2,h}^2.$$

Similarly, we can show that

$$\int_{\partial\Omega}\left(2u_{nn}u_n+\frac{\gamma_1}{h}u_n^2\right)ds\geqslant-\frac{\alpha}{3}|u|_{2,h}^2,$$

if constant $\gamma_1$ is suitably chosen. Therefore, we have proved

**Theorem 2.1.** *If $a_h(u, v)$ is uniformly $V_h$-elliptic and constants $\gamma$, $\gamma_1$ are suitably chosen, then the bilinear form $b(u, v)$ is also uniformly $V_h$-elliptic.*

In fact, from what we have proved,

$$b(u, u)=a_h(u, u)+c(u, u)-\int_{\partial\Omega}\left(2u_{nn}u_n+\frac{\gamma_1}{h}u_n^2\right)ds$$

$$\geqslant\alpha\|u\|_{2,h}^2-\frac{\alpha}{3}|u|_{2,h}^2-\frac{\alpha}{3}|u|_{2,h}^2\geqslant\frac{\alpha}{3}\|u\|_{2,h}^2.$$

**Remark.** For some finite element space $V_h$, it is possible and convenient to use $|u|_{2,h}$ as the norm of $V_h$, see, for instance, [2, p. 366]. If, instead of $\|u\|_{2,h}$, we use this norm in (2.10) and $a_h(\cdot,\cdot)$ is uniformly $V_h$-elliptic, it is obvious that Theorem 2.1 keeps valid with this norm.

**Theorem 2.2.** *There exists a constant $M$ independent of $u$, $v$ and $h$, such that for $\forall\, u, v\in V_h$,*

$$|b(u, v)|\leqslant Mh^{-2}\|u\|_{2,h}\|v\|_{2,h}.$$

*Proof.* Using Schwarz' inequality, we can easily see

$$|a_h(u, v)| \leqslant |u|_{2,h} |v|_{2,h}. \tag{2.11}$$

Since

$$(\tilde{u}_{nn})^2 \leqslant \frac{1}{2}((u_{nn}^+)^2 + (u_{nn}^-)^2),$$

$$[v_n]^2 \leqslant 2((v_n^+)^2 + (v_n^-)^2).$$

We have

$$\left|\int_\tau \tilde{u}_{nn}[v_n]ds\right| \leqslant \left(\int_\tau (\tilde{u}_{nn})^2 ds\right)^{\frac{1}{2}} \left(\int_\tau [v_n]^2 ds\right)^{\frac{1}{2}}$$

$$\leqslant \left(\int_\tau ((u_{nn}^+)^2 + (u_{nn}^-)^2) ds\right)^{\frac{1}{2}} \left(\int_\tau ((v_n^+)^2 + (v_n^-)^2) ds\right)^{\frac{1}{2}}$$

$$\leqslant (Ch^{-1}|u|_{2,\partial\tau}^2)^{\frac{1}{2}} (Ch^{-1}|v|_{1,\partial\tau}^2)^{\frac{1}{2}}, \tag{2.12}$$

where Lemma 2.1 is used in deriving the last inequality. Similary,

$$\frac{\gamma}{h}\left|\int_\tau [u_n][v_n]ds\right| \leqslant Ch^{-2}|u|_{1,\partial\tau}|v|_{1,\partial\tau}.$$

Taking the sum with respect to $\tau \in T_h$, we have

$$|c(u, v)| \leqslant Ch^{-2}\|u\|_{2,h}\|v\|_{2,h}.$$

Also, it is easy to show in the same way that

$$\left|\int_{\partial\Omega} \left(u_{nn}v_n + v_{nn}u_n + \frac{\gamma_1}{h} u_n v_n\right) ds\right| \leqslant Ch^{-2}\|u\|_{2,h}\|v\|_{2,h},$$

and the theorem follows.

**Corollary.** The variational problem (2.8) has a unique solution.

*Proof.* It is immediate from Theorems 2.1, 2.2 and the Lax-Milgram theorem.

## § 3. Error Estimate and Convergence

Let us first give the following lemma.

**Lemma 3.1.** *If $u \in H^{k+1}(\sigma)$ and $\Pi u$ is the interpolation of $u$ in $P_k(\sigma)$, then*

$$\int_{\partial\sigma} ((u - \Pi u)_{r,s})^2 ds \leqslant Ch^{2k-2i+1}|u|_{k+1,\sigma}^2, \quad r+s=i,\ i=0, 1, \cdots, k, \tag{3.1}$$

*where $(w)_{r,s} = \dfrac{\partial^{r+s} w}{\partial x_1^r \partial x_2^s}$ and constant $C$ is independent of $u$, $\sigma$ and $h$.*

The proof of the lemma is easy and so is omitted.

Throughout this section $a_h(\cdot, \cdot)$ is assumed to be uniformly $V_h$-elliptic and all the assumptions on $V_h$ in § 2 are valid. Now we can prove

**Theorem 3.1.** *If $u \in H^{k+1}(\Omega) \cap H_0^2(\Omega)$ and $u_h$ are the solutions of the problems (2.3) and (2.8) respectively, where $k \geqslant 2$ and $P_k(\sigma) \subset V_h(\sigma)$, then*

$$\|u - u_h\|_{2,h} \leqslant Ch^{k-2}|u|_{k+1,\Omega}.$$

*Proof.* Assuming $u \in H^4(\Omega)$ temporarily and using Green's formula on each element, we see for any $v \in V_h$

$$a_h(u, v) = (f, v)_0 - \sum_{\tau \in T_h} \int_\tau u_{nn}[v_n]ds + \int_{\partial\Omega} u_{nn}v_n ds. \tag{3.2}$$

In fact, (3.2) is still valid for $u \in H^l(\Omega)$, $l > 5/2$. The proof can be found in [1].

By the uniform $V_h$-ellipticity of $b(u, v)$, if $v \in V_h$,

$$\alpha\|u_h-v\|_{2,h}^2\leqslant b(u_h-v,\ u_h-v)$$
$$=b(u-v,\ u_h-v)+b(u_h,\ u_h-v)-b(u,\ u_h-v)$$
$$\leqslant|b(u-v,\ u_h-v)|+|(f,\ u_h-v)_0-b(u,\ u_h-v)|, \tag{3.3}$$

where we have used $b(u_h,\ u_h-v)=(f,\ u_h-v)_0$. It is seen from the trace theorem that when $u\in H^l(\Omega)$, $l\geqslant 3$,

$$u^+_{x_ix_j}=u^-_{x_ix_j}, \text{ a.e. on } \tau, \text{ for } \forall\tau\in T_h.$$

Thus $\tilde{u}_{x_ix_j}=u_{x_ix_j}$, $\tilde{u}_{nn}=u_{nn}$. Also, $[u_n]=0$. So

$$c(u,\ v)=\sum_{\tau\in T_h}\int_\tau u_{nn}[v_n]ds.$$

Then by (3.2),

$$b(u,\ v)=a_h(u,\ v)+c(u,\ v)-\int_{\partial\Omega}u_{nn}v_n\,ds=(f,\ v)_0$$

holds for $\forall v\in V_h$. Now (3.3) gives

$$\alpha\|u_h-v\|_{2,h}^2\leqslant|b(u-v,\ u_h-v)|. \tag{3.4}$$

Take $v$ to be $\Pi u$, the interpolation of $u$ in $V_h$. It is well known that

$$\|u-\Pi u\|_{2,h}\leqslant Ch^{k-1}|u|_{k+1,\Omega}, \tag{3.5}$$

and for $w\in V_h$,

$$|a_h(u-\Pi u,\ w)|\leqslant\|u-\Pi u\|_{2,h}\|w\|_{2,h}. \tag{3.6}$$

By Lemma 3.1,

$$\int_\tau(u-\Pi\tilde{u})_{nn}^2ds\leqslant Ch^{2k-3}|u|_{k+1,\delta\tau}^2,$$
$$\int_\tau(u-\Pi\tilde{u})_n^2ds\leqslant Ch^{2k-1}|u|_{k+1,\delta\tau}^2.$$

We see

$$\left|\int_\tau((u-\Pi\tilde{u})_{nn}[w_n]+\tilde{w}_{nn}[(u-\Pi u)_n])ds\right.$$
$$\leqslant Ch^{k-2}|u|_{k+1,\delta\tau}|w|_{1,\delta\tau}+Ch^{k-1}|w|_{2,\delta\tau}|u|_{k+1,\delta\tau}$$
$$\leqslant Ch^{k-2}|u|_{k+1,\delta\tau}\|w\|_{2,\delta\tau},$$
$$\left|\frac{\gamma}{h}\int_\tau[(u-\Pi u)_n][w_n]\,ds\right|\leqslant Ch^{k-2}|u|_{k+1,\delta\tau}|w|_{1,\delta\tau}.$$

Therefore

$$|c(u-\Pi u,\ w)|\leqslant Ch^{k-2}|u|_{k+1,\Omega}\|w\|_{2,h}. \tag{3.7}$$

Similarly,

$$\left|\int_{\partial\Omega}\left\{(u-\Pi u)_{nn}w_n+w_{nn}(u-\Pi u)_n+\frac{\gamma_1}{h}(u-\Pi u)_nw_n\right\}ds\right|$$
$$\leqslant Ch^{k-2}|u|_{k+1,\Omega}\|w\|_{2,h}. \tag{3.8}$$

Using (3.4)—(3.8), we have

$$\alpha\|u_h-\Pi u\|_{2,h}^2\leqslant Ch^{k-2}|u|_{k+1,\Omega}\|u_h-\Pi u\|_{2,h}.$$

Hence

$$\|u_h-\Pi u\|_{2,h}\leqslant Ch^{k-2}|u|_{k+1,\Omega}.$$

Then, by inequality

$$\|u-u_h\|_{2,h}\leqslant\|u-\Pi u\|_{2,h}+\|\Pi u-u_h\|_{2,h},$$

we get the conclusion of the theorem.

This theorem shows that if $k>2$, $\|u-u_h\|_{2,h}\to 0$, as $h\to 0$.

It is shown in [1, 3] and [8] that the convergence order of the finite element approximation given by the penalty method is $h^{(k-1)/2}$. We see that the compensation method is better if $k>3$.

The concept of weakly discontinuous finite element was given in [3] as follows. If for any $v\in V_h$,

$$\int_\tau [v]^2\,ds\leqslant Ch^3|v|^2_{2,\delta\tau}, \tag{3.9}$$

and

$$\int_\tau [v_{x_i}]^2\,ds\leqslant C_1h|v|^2_{2,\delta\tau},\quad i=1,\ 2 \tag{3.10}$$

hold for all $\tau\in T_h$, $V_h$ is a weakly discontinuous finite element space for fourth order equations. The constants $C$ and $C_1$ of (3.9) and (3.10) are independent of $v$ and $h$.

We can prove that if on each interelement side

1) there are at least two common interpolating points of the function, and

2) there is at least one common interpolating point of the normal derivative or the mean values of the normal derivatives of two adjacent elements are equal, i.e.

$$\int_\tau u_n^+\,ds=\int_\tau u_n^-\,ds,$$

(3.9) and (3.10) are then valid. We have

**Theorem 3.2.** *Keeping the assumptions of Theorem 3.1 and assuming that $V_h$ is a weakly discontinuous finite element space and if $v\in V_h$ is taken to be zero outside $\Omega$ and (3.10) is also valid on $\tau\subset\partial\Omega$, then we have*

$$\|u-u_h\|_{2,h}\leqslant Ch^{k-1}|u|_{k+1,\Omega}.$$

*Proof.* (3.10) shows, for $w\in V_h$,

$$\int_\tau [w_n]^2\,ds\leqslant Ch|w|^2_{2,\delta\tau}.$$

Hence

$$\left|\int_\tau (u-\Pi\tilde u)_{nn}[w_n]\,ds\right|\leqslant Ch^{k-1}|u|_{k+1,\delta\tau}|w|_{2,\delta\tau},$$

$$\left|\frac{\gamma}{h}\int_\tau [(u-\Pi u)_n]\,[w_n]\,ds\right|\leqslant Ch^{k-1}|u|_{k+1,\delta\tau}|w|_{2,\delta\tau}.$$

So we get

$$|c(u-\Pi u,\ w)|\leqslant Ch^{k-1}|u|_{k+1,\Omega}|w|_{2,h}. \tag{3.11}$$

If $\tau\subset\partial\Omega$,

$$\left|\int_\tau (u-\Pi u)_{nn}w_n\,ds\right|\leqslant Ch^{k-1}|u|_{k+1,\sigma}|w|_{2,\sigma},$$

where $\sigma\in\Omega_h$ is the element which has $\tau$ as a side. Thus

$$\left|\int_{\partial\Omega}\left\{(u-\Pi u)_{nn}w_n+w_{nn}(u-\Pi u)_n+\frac{\gamma_1}{h}(u-\Pi u)_nw_n\right\}ds\right| \leqslant Ch^{k-1}|u|_{k+1,\Omega}\|w\|_{2,h}. \tag{3.12}$$

Then (3.4)—(3.6), (3.11) and (3.12) give the conclusion.

Now, we see in Theorem 3.2 that the convergence order is the same with that of conforming elements which have $P_k(\sigma)\subset V_h(\sigma)$, if the assumptions on $V_h$ are valid. According to what we mentioned earlier, most of the "plate elements" in use satisfy the requirements.

## § 4. Plates with Other Boundary Conditions

If the plate considered is simply supported at the boundary, in the case of the plate being a polygon, the boundary condition is (cf. [4], p. 173):

$$u=u_{nn}=0, \quad (x_1, x_2)\in\partial\Omega, \tag{4.1}$$

instead of (2.2). The variational problem associated with the boundary value problem (2.1)—(4.1) is:

$$\text{Find } u\in V=H^2(\Omega)\cap H_0^1(\Omega), \text{ such that}$$

$$a(u, v)=(f, v)_0, \quad \forall v\in V. \tag{4.2}$$

It is easy to see that $a(u, v)$ is positive definite in $V$, i.e. for any $v\in V$, $v\neq 0$, $a(v, v)>0$. We see that $|v|_{2,\Omega}$ is also a norm on $V$. Thus (4.2) has a unique solution.

Let $V_h$ be the finite element space mentioned in § 2. Recall that we have assumed that

1) $V_h\subset C(\Omega)$, $P_k(\sigma)\subset V_h(\sigma)$, $\forall\sigma\in\Omega_h$;

2) $v|_{\partial\Omega}=0$, $\forall v\in V_h$.

In addition, we will assume that

3) there is at least one common point for each pair of adjacent elements, on which all first order partial derivatives of each $v\in V_h$ are continuous.

Now we can see that

$$v\to|v|_{2,h}=(a_h(v, v))^{\frac{1}{2}} \tag{4.3}$$

is also a norm of $V_h$. Actually, $|v|_{2,h}=0$ implies that $v_{x_1}$ and $v_{x_2}$ are constants in each element. Because they are continuous on some common points of any adjacent elements, they are constant in the whole domain $\Omega$, and then obviously $v=0$. Clearly, $a_h(u, v)$ is uniformly $V_h$-elliptic and bounded with respect to norm (4.3). Set

$$b_0(u, v)=a_h(u, v)+c(u, v),$$

i.e. delete the boundary integral term of $b(u, v)$. Take the solution of the problem:

$$\text{Find } u_h\in V_h, \text{ such that } b_0(u_h, v)=(f, v)_0, \quad \forall v\in V_h \tag{4.4}$$

as the approximation of $u$, the solution of (4.2). Similarly to what we have remarked in § 2, $b_0(u, v)$ is uniformly $V_h$-elliptic, if constant $\gamma$ of $c(u, v)$ is suitably chosen.

Moreover, we assume that

4) $V_h$ is a weakly discontinuous finite element space. Then we have, for $u, v\in V_h$,

$$\left|\int_\tau \tilde{u}_{nn}[v_n]\,ds\right|\leqslant\left(\int_\tau ((u_{nn}^+)^2+(u_{nn}^-)^2)\,ds\right)^{\frac{1}{2}}\left(\int_\tau [v_n]^2ds\right)^{\frac{1}{2}}$$

$$\leqslant(Ch^{-1}|u|_{2,\delta\tau}^2)^{\frac{1}{2}}(Ch|v|_{2,\delta\tau}^2)^{\frac{1}{2}}\leqslant C|u|_{2,\delta\tau}|v|_{2,\delta\tau}.$$

Here (3.10) has been used. So

$$|c(u, v)|\leqslant C|u|_{2,h}|v|_{2,h}$$

and hence

$$|b_0(u, v)|\leqslant C|u|_{2,h}|v|_{2,h}.$$

We have proved

**Theorem 4.1.** *If assumptions 1)—4) on $V_h$ are valid, the bilinear form $b_0(u, v)$ is uniformly $V_h$-elliptic and bounded with respect to norm (4.3) and problem (4.4) has a unique solution.*

Similarly to the proof of Theorems 3.1 and 3.2, we can prove

**Theorem 4.2.** *Keep the assumption of Theorem 4.1 and assume that $u$ is the solution of (4.2), $u_h$ is that of (4.4), $u\in H^{k+1}(\Omega)$, $k\geqslant 2$. We have an error estimate:*

$$|u-u_h|_{2,h}\leqslant Ch^{k-1}|u|_{k+1,\Omega}.$$

**Remark.** If we cannot verify inequality $\alpha\|v\|^2_{2,h}\leqslant a_h(v, v)$, all the discussion here may also be applied to plates with clamped boundary except that $b(u, v)$ should be used to replace $b_0(u, v)$.

Next, suppose that the plate considered is with the free boundary condition on part of its boundary, say, $\partial\Omega_1$, and clamped at $\partial\Omega_2$, simply supported at $\partial\Omega_3$, where

$$\partial\Omega_1\cup\partial\Omega_2\cup\partial\Omega_3=\partial\Omega.$$

To ensure the uniqueness of the solution, we assume that $\partial\Omega_2\cup\partial\Omega_3\neq\emptyset$, and that $\partial\Omega_3$ contains several segments not lying on a straight line if $\partial\Omega_2=\emptyset$. The free boundary condition can be expressed as (see [4], p. 171)

$$\begin{gathered}\nu\Delta u+(1-\nu)u_{nn}=0,\\ (\Delta u)_{n}-(1-\nu)\frac{\partial}{\partial t}\{(u_{x_1x_1}-u_{x_2x_2})n_1n_2+u_{x_1x_2}(n_2^2-n_1^2)\}=0,\\ \text{for } (x_1, x_2)\in\partial\Omega_1,\end{gathered}\tag{4.5}$$

where $0<\nu<\frac{1}{2}$ is Poisson's coefficient, $n_i=\cos(n, x_i)$, $i=1, 2$ and $t$ is the tangential vector of $\partial\Omega$. Set

$$V_1=\{u\in H^2(\Omega);\ u=0 \text{ on } \partial\Omega_2\cup\partial\Omega_3,\ u_n=0 \text{ on } \partial\Omega_2\}$$

and

$$a_1(u, v)=\int_\Omega\{\Delta u\Delta v+(1-\nu)(2u_{x_1x_2}v_{x_1x_2}-u_{x_1x_1}v_{x_2x_2}-u_{x_2x_2}v_{x_1x_1})\}dx.$$

The corresponding variational problem is

$$\text{Find } u\in V_1, \text{ such that } a_1(u, v)=(f, v)_0,\quad \forall v\in V_1.\tag{4.6}$$

Suppose $V_h$ satisfies assumptions 1), 3) and 4) and an additional assumption

2′) $\forall v\in V_h$, $v=0$ on $\partial\Omega_2\cup\partial\Omega_3$, where $\partial\Omega_2\cup\partial\Omega_3$ is assumed to contain several segments not lying on a straight line.

Put, for $u, v\in V_h\cup V_1$,

$$a_{1h}(u, v)=\sum_{\sigma\in\Omega_h}\int_\sigma\{\Delta u\Delta v+(1-\nu)(2u_{x_1x_2}v_{x_1x_2}-u_{x_1x_1}v_{x_2x_2}-u_{x_2x_2}v_{x_1x_1})\}dx,$$

$$c_1(u, v)=\sum_{\tau\in T_h}\int_\tau\Big\{(\nu\widetilde{\Delta u}+(1-\nu)\tilde{u}_{nn})[v_n]-(\nu\widetilde{\Delta v}+(1-\nu)\tilde{v}_{nn})[u_n]+\frac{\gamma}{h}[u_n][v_n]\Big\}ds,$$

and

$$b_1(u, v) = a_{1h}(u, v) + c_1(u, v) - \int_{\partial\Omega_1} \Big\{ (\nu\Delta u + (1-\nu)u_{nn})v_n + (\nu\Delta v + (1-\nu)v_{nn})u_n + \frac{\gamma_1}{h} u_n v_n \Big\} ds.$$

It is not difficult to verify the uniform $V_h$-ellipticity and boundedness of $b_1(u, v)$ with respect to norm $|u|_{2,h}$. In fact,

$$a_{1h}(u, u) = \nu|\Delta u|_{0,h}^2 + (1-\nu)|u|_{2,h}^2 \geqslant \frac{1}{2}|u|_{2,h}^2$$

gives the ellipticity. Therefore the problem

$$\text{Find } u_h \in V_h, \text{ such that } b_1(u_h, v) = (f, v)_0, \quad \forall v \in V_h \tag{4.7}$$

is unisolvent.

For $u \in H^3(D)$, $v \in H^2(D)$, $D \subset R^2$ being a domain, the following formula holds

$$\int_D (2u_{x_1x_1}v_{x_1x_2} - u_{x_1x_1}v_{x_2x_2} - u_{x_2x_2}v_{x_1x_1})\, dx = \int_{\partial D} (-u_{tt}v_n + u_{tn}v_t) ds. \tag{4.8}$$

If $u \in H^4(\Omega)$ is the solution of (4.6), using Green's formula and (4.8) on $\sigma$, we get

$$\int_\sigma \{\Delta u \Delta v + (1-\nu)(2u_{x_1x_2}v_{x_1x_2} - u_{x_1x_1}v_{x_2x_2} - u_{x_2x_2}v_{x_1x_1})\} dx$$
$$= \int_\sigma \Delta^2 uv\, dx + \int_{\partial\sigma} (\nu\Delta u + (1-\nu)u_{nn})v_n ds - \int_{\partial\sigma} ((\Delta u)_n + (1-\nu)u_{ntt})v\, ds. \tag{4.9}$$

Notice

$$u_{nt} = -(u_{x_1x_1} - u_{x_2x_2})n_1n_2 - u_{x_1x_2}(n_2^2 - n_1^2).$$

Then taking the sum on both sides of (4.9) with respect to $\sigma \in \Omega_h$, we get, for $v \in V_h$,

$$a_{1h}(u, v) = (f, v)_0 - \sum_{\tau \in T_h} \int_\tau (\nu\Delta u + (1-\nu)u_{nn})[v_n] ds + \int_{\partial\Omega_1} (\nu\Delta u + (1-\nu)u_{nn})v_n ds. \tag{4.10}$$

Here we have used the boundary condition (4.5) on $\partial\Omega_1$ and condition (4.1) on $\partial\Omega_3$ and $u = u_n = 0$ on $\partial\Omega_2$. Since $u \in H^4(\Omega)$, (4.10) shows

$$b_1(u, v) = (f, v)_0.$$

Then, in line with the proof of Theorems 3.1, 3.2, we can prove

**Theorem 4.3.** If $u$ and $u_h$ are the solutions of (4.6) and (4.7) respectively, $u \in H^{k+1}(\Omega)$, $k \geqslant 3$, and $V_h$ satisfies assumptions 1), 2′), 3) and 4), we have an error estimate:

$$|u - u_h|_{2,h} \leqslant Ch^{k-1}|u|_{k+1,\Omega}.$$

The error estimates given by Theorems 4.2 and 4.3 are in terms of norm $|\cdot|_{2,h}$, but one usually wants to have estimates in norms $\|\cdot\|_{0,h}$, $\|\cdot\|_{1,h}$, etc. To accomplish this purpose, we can use the duality argument and an interpolation inequality for the derivatives given in [10]. Finally, for both simply supported plates and plates with partial free boundary, we have

$$\|u - u_h\|_{0,h} \leqslant Ch^k|u|_{k+1,\Omega}, \tag{4.11}$$

$$\|u - u_h\|_{1,h} \leqslant Ch^{k-1}|u|_{k+1,\Omega}. \tag{4.12}$$

Combine Theorem 4.2, or 4.3 with (4.11) and (4.12). We get

$$\|u-u_h\|_{2,h}\leqslant Ch^{k-1}|u|_{k+1,\Omega} \tag{4.13}$$

for both cases. Besides, as we mentioned in the remark of this section earlier, for clamped plates, if $|u|_{2,h}$ is used as the norm, estimates (4.11)—(4.13) are also valid in the case.

## § 5. Some Examples

1) Adini's rectangular element. It is easy to see from § 3 that $V_h$ is a weakly discontinuous finite element space. It has been proved in [2] that for the tolerance method, one has

$$|u-u_h|_{2,h}\leqslant Ch\|u\|_{3,\Omega}.$$

But for the compensation method, we have

$$|u-u_h|_{2,h}\leqslant Ch^2|u|_{4,\Omega}.$$

2) Fraeijs de Veubeke's triangular element. The degrees of freedom of the element are function values on the three vertices and the barycenter, and the normal derivatives on the two Gaussian points of each side, $k=3$. [5] proved

$$|u-u_h|_{2,h}\leqslant Ch(|u|_{3,\Omega}+h|u|_{4,\Omega})$$

for the tolerance method. The compensation method will have

$$|u-u_h|_{2,h}\leqslant Ch^2|u|_{4,\Omega}.$$

3) It was proved in [1] that using the penalty method for the complete 3 degree triangle element (the degrees of freedom are function values and two first-order partial derivatives on each vertex and the function value on the barycenter), the convergence order is $O(h)$. But that of the compensation method will be $O(h^2)$.

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