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ALGEBRAIC CRITERION OF CONSISTENCY FOR GENERAL LINEAR METHODS OF ORDINARY DIFFERENTIAL EQUATIONS*

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§ 1. Introduction

Consider the initial value problem of ordinary differential equations

$$\begin{cases} \dfrac{dy}{dx}=f(y), \\ y(x_0)=y_0, \end{cases} \tag{1}$$

where $x\in R$, $y\in R^m$. In unifying the theory of various linear methods, Burrage and Butcher [1] presented the following general linear methods:

$$\begin{cases} Y_i^{(n)}=h\sum_{j=1}^{s}c_{ij}^{11}f(Y_j^{(n)})+\sum_{j=1}^{k}c_{ij}^{12}y_j^{(n-1)}, & i=1, 2, \cdots, s, \\ y_i^{(n)}=h\sum_{j=1}^{s}c_{ij}^{21}f(Y_j^{(n)})+\sum_{j=1}^{k}c_{ij}^{22}y_j^{(n-1)}, & i=1, 2, \cdots, k. \end{cases} \tag{2}$$

There are s internal vectors $Y_1^{(n)}, \cdots, Y_s^{(n)}$ and k external vectors $y_1^{(n)}, \cdots, y_k^{(n)}$, respectively. For the matrix-vector form, set

$$Y^{(n)}=Y_1^{(n)}\oplus Y_2^{(n)}\oplus\cdots\oplus Y_s^{(n)},$$
$$y^{(n)}=y_1^{(n)}\oplus y_2^{(n)}\oplus\cdots\oplus y_k^{(n)},$$
$$F(Y^{(n)})=f(Y_1^{(n)})\oplus f(Y_2^{(n)})\oplus\cdots\oplus f(Y_s^{(n)});$$
$$C_{11}=(c_{ij}^{11})_{s\times s},\ C_{12}=(c_{ij}^{12})_{s\times k},\ C_{21}=(c_{ij}^{21})_{k\times s},\ C_{22}=(c_{ij}^{22})_{k\times k};$$
$$C=\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},\quad [C]=\begin{pmatrix} C_{11}\otimes I_s & C_{12}\otimes I_k \\ C_{21}\otimes I_s & C_{22}\otimes I_k \end{pmatrix},$$

where I's are unit matrices. By means of the partitioned matrix C, the general linear methods (2) are expressed as

$$\begin{pmatrix} Y^{(n)} \\ y^{(n)} \end{pmatrix}=[C]\begin{pmatrix} hF(Y^{(n)}) \\ y^{(n-1)} \end{pmatrix}. \tag{3}$$

Burrage and Butcher [1] defined that if there exist vectors $u, v\in R^k$ such that

$$C_{12}u=e,\ C_{22}u=u, \tag{4}$$
$$C_{21}e+C_{22}v=u+v, \tag{5}$$

then the method (3) is said to be consistent, where $e=(1\ 1\ \cdots\ 1)^T$.

There are two questions: First, three submatrices C_{12}, C_{21} and C_{22} are constrained in the consistency conditions (4) and (5); and what about the submatrix

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C_{11}? Second, when the method (3) is given, how to determine the vectors u and v, and how to discuss the consistency of order $p\geqslant 1$?

Frank, Schneid and Ueberhuber [2] defined the so called B-consistency for the general linear methods. The method (3) is B-consistent of order $p\geqslant 1$, if the numerical solution ξ obtained one step forward from $y(x)$ (steplength h) and the exact solution $y(x+h)$ of problem (1) satisfy

$$\|\xi-y(x+h)\|\leqslant Dh^{p+1},\quad h\in(0,\,h_0],\tag{6}$$

where D and h_0 depend only on the logarithmic norm of $\dfrac{\partial f}{\partial y}$ and other derivatives of f.

This consistency condition seems to be strengthened in order to gain the B-convergence. And no algebraic criterion of the consistency is given.

In this paper, the necessary and sufficient conditions of consistency of order $p\geqslant 1$ for the general linear method (3) are presented. They have very wide application in various linear methods.

§ 2. Main Result

According to the feature of general linear method (3), the internal vectors $Y_i^{(n)}$ must be in the domain of f, so they usually are approximations of the exact solution $y(x)$ of problem (1) at the point x_0+nh+w_ih:

$$Y_i^{(n)}\sim\eta_i^{(n)}=y(x_0+nh+w_ih),\quad i=1,\,2,\,\cdots,\,s.\tag{7}$$

The external vectors $y_i^{(n)}$ are relatively free; they may be expressed in a rather general form:

$$y_i^{(n)}\sim\zeta_i^{(n)}=\alpha_iy(x_0+nh+\gamma_ih)+\beta_ihy'(x_0+nh+\delta_ih),\quad i=1,\,2,\,\cdots,\,k.\tag{8}$$

When the method (3) is given, the parameters w_i, α_i, β_i, γ_i and δ_i can be determined. Particularly, $\alpha_i^2+\beta_i^2\neq 0$ and $\alpha_i\cdot\beta_i=0$. Set

$$H^{(n)}=\eta_1^{(n)}\oplus\eta_2^{(n)}\oplus\cdots\oplus\eta_s^{(n)},\quad Z^{(n)}=\zeta_1^{(n)}\oplus\zeta_2^{(n)}\oplus\cdots\oplus\zeta_k^{(n)}.\tag{9}$$

As usual,

$$\zeta_i^{(n-1)}=\alpha_iy(x_0+(n-1)h+\gamma_ih)+\beta_ihy'(x_0+(n-1)h+\delta_ih),\quad i=1,\,2,\,\cdots,\,k.\tag{10}$$

Definition. *For the general linear method* (3), *if*

$$\left\|\begin{pmatrix}H^{(n)}\\ Z^{(n)}\end{pmatrix}-[C]\begin{pmatrix}hF(H^{(n)})\\ Z^{(n-1)}\end{pmatrix}\right\|=O(h^{p+1}),\tag{11}$$

then it is said to be consistent of order p.

Obviously, this is a measure about the local discretization error of the scheme (3), and it coincides with the proper sense of the concept "consistency". The key to the establishment of consistency definition (11) lies in expressions (7) and (8) of the internal and external vectors.

Before formulating the algebraic criterion of consistency for general linear methods, we set the following symbols.

For the vectors

$$Q=(q_1, q_2, \cdots, q_l)^T, \ R=(r_1, r_2, \cdots, r_l)^T,$$

define a new vector whose components equal the product of the corresponding components of Q and R

$$Q^*R=(q_1r_1, q_2r_2, \cdots, q_lr_l)^T.$$

Furthermore

$$Q^0=e, \ Q^1=Q,$$

and

$$Q^\nu=Q^*Q^{\nu-1}=(q_1^\nu, q_2^\nu, \cdots, q_l^\nu)^T, \quad \nu=1, 2, \cdots.$$

According to (7) and (8), set vectors

$$\begin{gathered} w=(w_1, w_2, \cdots, w_s)^T; \\ \alpha=(\alpha_1, \alpha_2, \cdots, \alpha_k)^T, \quad \beta=(\beta_1, \beta_2, \cdots, \beta_k)^T, \\ \gamma=(\gamma_1, \gamma_2, \cdots, \gamma_k)^T, \quad \delta=(\delta_1, \delta_2, \cdots, \delta_k)^T. \end{gathered} \tag{12}$$

Theorem 1. *The necessary and sufficient conditions of consistency of order p for the general linear method* (3) *are*

$$C_{12}\alpha=e, \quad C_{22}\alpha=\alpha; \tag{13}$$

and

$$\begin{aligned} &C_{11}\left\{\frac{w^{\nu-1}}{(\nu-1)!}\right\}+C_{12}\left\{\alpha*\frac{(\gamma-e)^\nu}{\nu!}+\beta*\frac{(\delta-e)^{\nu-1}}{(\nu-1)!}\right\}=\frac{w^\nu}{\nu!}, \\ &C_{21}\left\{\frac{w^{\nu-1}}{(\nu-1)!}\right\}+C_{22}\left\{\alpha*\frac{(\gamma-e)^\nu}{\nu!}+\beta*\frac{(\delta-e)^{\nu-1}}{(\nu-1)!}\right\} \\ &\quad=\alpha*\frac{\gamma^\nu}{\nu!}+\beta*\frac{\delta^{\nu-1}}{(\nu-1)!}, \quad \nu=1, 2, \cdots, p. \end{aligned} \tag{14}$$

Proof. By substitution into (3) with

$$\begin{aligned} \eta_i=&y(x_n)+w_ihy'(x_n)+w_i^2\frac{h^2}{2}y''(x_n)+\cdots+w_i^p\frac{h^p}{p!}y^{(p)}(x_n) \\ &+w_i^{p+1}\frac{h^{p+1}}{(p+1)!}y^{(p+1)}(x_n)+o(h^{p+1}), \end{aligned}$$

$$\begin{aligned} \zeta_i=&\alpha_iy(x_n)+(\alpha_i\cdot\gamma_i+\beta_i)hy'(x_n)+\left(\alpha_i\frac{\gamma_i^2}{2}+\beta_i\cdot\delta_i\right)h^2y'(x_n) \\ &+\cdots+\left(\alpha_i\frac{\gamma_i^p}{p!}+\beta_i\frac{\delta_i^{p-1}}{(p-1)!}\right)h^py^{(p)}(x_n) \\ &+\left(\alpha_i\frac{\gamma_i^{p+1}}{(p+1)!}+\beta_i\frac{\delta_i^p}{p!}\right)h^{p+1}y^{(p+1)}(x_n)+o(h^{p+1}), \end{aligned}$$

and

$$\begin{aligned} hf(\eta_i)=&hy'(x_n)+w_ih^2y''(x_n)+\cdots+\frac{w_i^{p-1}}{(p-1)!}h^py^{(p)}(x_n) \\ &+\frac{w_i^p}{p!}h^{p+1}y^{(p+1)}(x_n)+o(h^{p+1}), \end{aligned}$$

it follows that

$$\begin{pmatrix} e \\ \alpha \end{pmatrix} \otimes y(x_n) + h \begin{pmatrix} w \\ \alpha * \gamma + \beta \end{pmatrix} \otimes y'(x_n) + h^2 \begin{pmatrix} \frac{w^2}{2} \\ \alpha * \frac{\gamma^2}{2} + \beta * \delta \end{pmatrix} \otimes y''(x_n) + \cdots$$

$$+ h^p \begin{pmatrix} \frac{w^p}{p!} \\ \alpha * \frac{\gamma^p}{p!} + \beta * \frac{\delta^{p-1}}{(p-1)!} \end{pmatrix} \otimes y^{(p)}(x_n)$$

$$+ h^{p+1} \begin{pmatrix} \frac{w^{p+1}}{(p+1)!} \\ \alpha * \frac{\gamma^{p+1}}{(p+1)!} + \beta * \frac{\delta^p}{p!} \end{pmatrix} \otimes y^{(p+1)}(x_n) + o(h^{p+1})$$

$$= [C] \Bigg\{ \begin{pmatrix} \theta \\ \alpha \end{pmatrix} \otimes y(x_n) + h \begin{pmatrix} e \\ \alpha * (\gamma - e) + \beta \end{pmatrix} \otimes y'(x_n)$$

$$+ h^2 \begin{pmatrix} w \\ \alpha * \frac{(\gamma - e)^2}{2} + \beta * (\delta - e) \end{pmatrix} \otimes y''(x_n) + \cdots$$

$$+ h^p \begin{pmatrix} \frac{w^{p-1}}{(p-1)!} \\ \alpha * \frac{(\gamma - e)^p}{p!} + \beta * \frac{(\delta - e)^{p-1}}{(p-1)!} \end{pmatrix} \otimes y^{(p)}(x_n)$$

$$+ h^{p+1} \begin{pmatrix} \frac{w^p}{p!} \\ \alpha * \frac{(\gamma - e)^{p+1}}{(p+1)!} + \beta * \frac{(\delta - e)^p}{p!} \end{pmatrix} \otimes y^{(p+1)}(x_n) + o(h^{p+1}) \Bigg\},$$

where the null vector $\theta \in R^s$. Therefore, it is consistent of order p iff (13) and (14) are satisfied. Then, the local discretization error is written as

$$\begin{pmatrix} H^{(n)} \\ Z^{(n)} \end{pmatrix} - [C] \begin{pmatrix} hF(H^{(n)}) \\ Z^{(n-1)} \end{pmatrix} = h^{p+1} \Bigg\{ \begin{pmatrix} \frac{w^{p+1}}{(p+1)!} \\ \alpha * \frac{\gamma^{p+1}}{(p+1)!} + \beta * \frac{\delta^p}{p!} \end{pmatrix}$$

$$- [C] \begin{pmatrix} \frac{w^p}{p!} \\ \alpha * \frac{(\gamma - e)^{p+1}}{(p+1)!} + \beta * \frac{(\delta - e)^p}{p!} \end{pmatrix} \Bigg\} \otimes y^{(p+1)}(x_n) + o(h^{p+1}). \tag{15}$$

This completes the proof.

Remark. As usual, the consistency means the case $p=1$. From the above theorem, the following conditions are necessary and sufficient:

$$C_{12}\alpha = e, \quad C_{22}\alpha = \alpha, \tag{16}$$

$$C_{11}e + C_{12}\{\alpha * \gamma + \beta\} = e + w, \tag{17}$$

$$C_{21}e + C_{22}\{\alpha * \gamma + \beta\} = \alpha + \{\alpha * \gamma + \beta\}. \tag{18}$$

Let $u = \alpha$, $v = \alpha * \gamma + \beta$, then condition (16) is just (4), and condition (18) is just (5).

Next, we shall show that (17) is independent of (16) and (18).

§ 3. Direct Applications

By means of the established algebraic criterions (12) to (14), we discuss the consistency for given linear methods.

1° Linear K-step methods

$$\sum_{i=0}^{k} a_i y_{n+i} = h \sum_{i=0}^{k} b_i f(y_{n+i}), \tag{19}$$

where $a_k \neq 0$, $a_0^2 + b_0^2 \neq 0$. The characteristic polynomials

$$\rho(t) = \sum_{i=0}^{k} a_i t^i, \quad \sigma(t) = \sum_{i=0}^{k} b_i t^i$$

have no common factor. At present, only one internal vector $Y^{(n)}$ is needed. And $2k$ external vectors needed are as follows:

$$y^{(n)} = y_{n+k} \oplus y_{n+k-1} \oplus \cdots \oplus y_{n+1} \oplus hf(y_{n+k}) \oplus \cdots \oplus hf(y_{n+1}).$$

Then the partitioned matrix form (3) is constructed with

$$C_{11} = \left(\frac{b_k}{a_k}\right), \quad C_{12} = \left(\frac{-a_{k-1}}{a_k} \cdots \frac{-a_0}{a_k}\ \frac{b_{k-1}}{a_k} \cdots \frac{b_0}{a_k}\right),$$

$$C_{21} = \begin{pmatrix} \frac{b_k}{a_k} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad C_{22} = \begin{pmatrix} \frac{-a_{k-1}}{a_k} & \frac{-a_{k-2}}{a_k} & \cdots & \frac{-a_1}{a_k} & \frac{-a_0}{a_k} & \frac{b_{k-1}}{a_k} & \frac{b_{k-2}}{a_k} & \cdots & \frac{b_1}{a_k} & \frac{b_0}{a_k} \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 1 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & & 0 & 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

From scheme (19) it follows that

$$w = (k), \quad e = (1);$$

$$\alpha = (1, 1, \cdots 1, 0, \cdots 0)^T, \ \beta = (0, \cdots 0, 1, 1, \cdots 1)^T,$$

$$\gamma = (k, k-1, \cdots 1, 0, \cdots 0)^T, \ \delta = (0, \cdots 0, k, k-1, \cdots 1)^T.$$

Thus, condition (16) requires

$$\rho(1) = 0.$$

Conditions (17) and (18) lead to

$$\rho'(1) = \sigma(1).$$

Clearly, these are just the consistency conditions for linear multistep methods in the proper sense.

Furthermore, we examine the consistency conditions of order 2. When $\nu = 2$, (14) requires

$$\begin{cases} C_{11}w+C_{12}\left\{\alpha*\dfrac{(\gamma-e)^2}{2}+\beta*(\delta-e)\right\}=\dfrac{w^2}{2}, \\ C_{21}w+C_{22}\left\{\alpha*\dfrac{(\gamma-e)^2}{2}+\beta*(\delta-e)\right\}=\alpha*\dfrac{\gamma^2}{2}+\beta*\delta, \end{cases} \tag{20}$$

that is

$$\rho''(1)+\rho'(1)=2\sigma'(1).$$

The well-known BDF, as a special case, are included in the above linear multistep methods.

2° Hybrid methods ($P_\nu E\ P_k E\ C_H E$):

$$\begin{cases} y_{n+\nu}+\sum_{i=0}^{k-1}\bar{a}_i y_{n+i}=h\sum_{i=0}^{k-1}\bar{b}_i f(y_{n+i}), & (P_\nu) \\ y^*_{n+k}+\sum_{i=0}^{k-1}a^*_i y_{n+i}=h\sum_{i=0}^{k-1}b^*_i f(y_{n+i}), & (P_k) \\ y_{n+k}+\sum_{i=0}^{k-1}a_i y_{n+i}=h\sum_{i=0}^{k-1}b_i f(y_{n+i})+hb_k f(y^*_{n+k})+hb_\nu f(y_{n+\nu}). & (C_H) \end{cases}$$

There are three pairs of characteristic functions:

$$\bar{\rho}(t)=t^\nu+\sum_{i=0}^{k-1}\bar{a}_i t^i,\quad \bar{\sigma}(t)=\sum_{i=0}^{k-1}\bar{b}_i t^i,$$

$$\rho^*(t)=t^k+\sum_{i=0}^{k-1}a^*_i t^i,\quad \sigma^*(t)=\sum_{i=0}^{k-1}b^*_i t^i,$$

$$\rho(t)=t^k+\sum_{i=0}^{k-1}a_i t^i,\quad \sigma(t)=b_k t^k+b_\nu t^\nu+\sum_{i=0}^{k-1}b_i t^i.$$

Now, 3 internal vectors and $2k$ external vectors needed are as follows:

$$Y^{(n)}=Y_1^{(n)}\oplus Y_2^{(n)}\oplus Y_3^{(n)},$$

$$y^{(n)}=y_{n+k}\oplus y_{n+k-1}\oplus\cdots\oplus y_{n+1}\oplus hf(y_{n+k})\oplus hf(y_{n+k-1})\oplus\cdots\oplus hf(y_{n+1}).$$

In the partitioned matrix expression (3),

$$C_{11}=\begin{pmatrix}0&0&0\\0&0&0\\b_\nu&b_k&0\end{pmatrix},\quad C_{12}=\begin{pmatrix}-\bar{a}_{k-1}&\cdots&-\bar{a}_0&\bar{b}_{k-1}&\cdots&\bar{b}_0\\-a^*_{k-1}&\cdots&-a^*_0&b^*_{k-1}&\cdots&b^*_0\\-a_{k-1}&\cdots&-a_0&b_{k-1}&\cdots&b_0\end{pmatrix},$$

$$C_{21}=\begin{pmatrix}b_\nu&b_k&0\\0&0&0\\\vdots&\vdots&\vdots\\0&0&0\\0&0&1\\0&0&0\\\vdots&\vdots&\vdots\\0&0&0\end{pmatrix},\quad C_{22}=\begin{pmatrix}-a_{k-1}&\cdots&-a_1&-a_0&b_{k-1}&\cdots&b_1&b_0\\1&\cdots&0&0&0&\cdots&0&0\\\vdots&&&&&&&\vdots\\0&\cdots&1&0&0&\cdots&0&0\\0&\cdots&0&0&0&\cdots&0&0\\0&\cdots&0&0&1&\cdots&0&0\\\vdots&&&&&&&\vdots\\0&\cdots&0&0&0&\cdots&1&0\end{pmatrix}.$$

The following vectors are provided immediately by the schemes (P_ν), (P_k) and (C_H):

$$w=(\nu,\ k,\ k)^T,$$

$$\alpha=(1,\ 1,\ \cdots\ 1,\ 0,\ \cdots\ 0)^T,\ \beta=(0,\ \cdots\ 0,\ 1,\ 1,\ \cdots\ 1)^T,$$

$$\gamma=(k,\ k-1,\ \cdots 1,\ 0,\ \cdots 0)^T,\ \delta=(0,\ \cdots 0,\ k,\ k-1,\ \cdots 1)^T.$$

Condition (16) is just

$$\bar{\rho}(1)=0,\ \rho^*(1)=0,\ \rho(1)=0.$$

Remarkably, condition (17) leads to

$$\bar{\rho}'(1)=\sigma(1),\ \rho^{*\prime}(1)=\sigma^*(1),\ \rho'(1)=\sigma(1);$$

but condition (18) requires only

$$\rho'(1)=\sigma(1).$$

It is shown that condition (17) is independent of (16) and (18) indeed.

The predictor-corrector methods (PECE) may be handled as a special case of the above hybrid methods.

§ 4. Remarks

With respect to the expressions of $Y^{(n)}$ and $y^{(n-1)}$, we make a few remarks for some particular cases.

3° One-leg methods. For any linear k-step method

$$\sum_{i=0}^{k} a_i y_{n+i}=h\sum_{i=0}^{k} b_i f(y_{n+i}),$$

without loss of generality assume

$$\sum_{i=0}^{k} b_i=1.$$

Its corresponding one-leg method is

$$\sum_{i=0}^{k} a_i y_{n+i}=hf\left(\sum_{i=0}^{k} b_i y_{n+i}\right). \tag{21}$$

Now, the sole internal vector should be expressed as

$$Y^{(n)}\sim\sum_{i=0}^{k} b_i y(x_n+ih)$$
$$=y(x_n)+h\left(\sum_{i=0}^{k} ib_i\right)y'(x_n)+\frac{h^2}{2}\left(\sum_{i=0}^{k} i^2 b_i\right)y''(x_n)+\cdots.$$

In this case, expression (7) is replaced by the more general one:

$$Y_i^{(n)}\sim y(x_n)+hw_i^{[1]}y'(x_n)+\frac{h^2}{2}w_i^{[2]}y''(x_n)+\cdots. \tag{7*}$$

Accordingly, in (14), (15) etc. w^ν are replaced by $w^{[\nu]}$:

$$w^{[\nu]}=(w_1^{[\nu]},\ w_2^{[\nu]},\ \cdots\ w_s^{[\nu]})^T.\quad \nu=1,\ 2,\ \cdots,\ p. \tag{22}$$

For the one-leg method (21),

$$w^{[1]}=\sum_{i=0}^{k} ib_i=\sigma'(1),\quad w^{[2]}=\sum_{i=0}^{k} i^2 b_i.$$

The formula (21) can be written as

$$Y^{(n)}=\sum_{i=0}^{k-1}\left(b_i-\frac{b_k}{a_k}a_i\right)y_{n+i}+\frac{b_k}{a_k}hf(Y^{(n)}).$$

Then set k external vector as follows:

$$y^{(n)}=y_{n+k}\oplus y_{n+k-1}\oplus\cdots\oplus y_{n+1}.$$

In the partitioned matrix form (3),

$$C_{11}=\left(\frac{b_k}{a_k}\right),\quad C_{12}=\left(b_{k-1}-\frac{b_k}{a_k}a_{k-1}\cdots b_0-\frac{b_k}{a_k}a_0\right),$$

$$C_{21}=\begin{pmatrix}\frac{1}{a_k}\\0\\\vdots\\0\end{pmatrix},\quad C_{22}=\begin{pmatrix}-\frac{a_{k-1}}{a_k} & -\frac{a_{k-2}}{a_k} & \cdots & -\frac{a_1}{a_k} & -\frac{a_0}{a_k}\\ 1 & 0 & & 0 & 0\\ \vdots & & & & \vdots\\ 0 & 0 & \cdots & 1 & 0\end{pmatrix}.$$

Clearly,

$$\alpha=(1,1,\cdots 1)^T,\ \gamma=(k,k-1,\cdots 1)^T,\ \beta=\delta=\theta.$$

Thus, condition (16) requires $\rho(1)=0$, and (17) and (18) lead to $\rho'(1)=1$. These are usual consistency conditions. Under the modified expressions (7*) and (22), from (20) it follows that second order consistency requires $\rho''(1)+\rho'(1)=2\sigma'(1)$.

4° s-stage Runge-Kutta methods

$$\begin{cases}Y_i^{(n)}=y_{n-1}+h\sum_{j=1}^{s}a_{ij}f(Y_j^{(n)}), & i=1,2,\cdots,s,\\ y_n=y_{n-1}+h\sum_{j=1}^{s}b_jf(Y_j^{(n)}).\end{cases}\tag{23}$$

Now, s internal vectors are ready-made:

$$Y^{(n)}=Y_1^{(n)}\oplus Y_2^{(n)}\oplus\cdots\oplus Y_s^{(n)}.$$

And only one external vector $y^{(n)}$ is needed. Then, in the partitioned matrix form (3),

$$C_{11}=\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1s}\\ a_{21} & a_{22} & \cdots & a_{2s}\\ \vdots & & & \vdots\\ a_{s1} & a_{s2} & \cdots & a_{ss}\end{pmatrix},\quad C_{12}=\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix},$$

$$C_{21}=(b_1,b_2,\cdots,b_s),\quad C_{22}=(1).$$

According to the usual expression of Runge-Kutta methods, set $A=C_{11}$ and

$$c_i=\sum_{j=1}^{s}a_{ij},\quad i=1,2,\cdots,s.$$

Thus, we are able to write

$$w=Ae=(c_1,c_2,\cdots,c_s)^T\tag{24}$$

and

$$\alpha=\gamma=(1),\quad \beta=\delta=(0).$$

In this case, condition (16) is trivial. (16) and (18) lead to the following consistency condition for Runge-Kutta methods

$$\sum_{j=1}^{s}b_j=1.\tag{25}$$

Furthermore, we examine the fourth order consistency conditions. By (20), when $\nu=2$ we find

$$\sum_{j=1}^{s} b_j c_j = \frac{1}{2}, \tag{26}$$

$$\sum_{j=1}^{s} a_{ij} c_j = \frac{1}{2} c_i^2, \quad i=1, 2, \cdots, s. \tag{27}$$

Analogously, when $\nu=3$ we have

$$\sum_{j=1}^{s} b_j c_j^2 = \frac{1}{3}, \tag{28}$$

$$\sum_{j=1}^{s} a_{ij} c_j^2 = \frac{1}{3} c_i^3, \quad i=1, 2, \cdots, s. \tag{29}$$

Finally, when $\nu=4$ we get

$$\sum_{j=1}^{s} b_j c_j^3 = \frac{1}{4}, \tag{30}$$

$$\sum_{j=1}^{s} a_{ij} c_j^3 = \frac{1}{4} c_i^4, \quad i=1, 2, \cdots, s. \tag{31}$$

In brief, the fourth order consistency conditions for Runge–Kutta methods are (24)—(31).

It is easy to verify that $\sum_{i,j=1}^{s} b_i a_{ij} c_j = \frac{1}{6}$ can be derived from (27) and (28); $\sum_{i,j=1}^{s} b_i c_i a_{ij} c_j = \frac{1}{8}$ from (26), (27) and (29), $\sum_{i,j=1}^{s} b_i a_{ij} c_j^2 = \frac{1}{12}$ from (29) and (30); and $\sum_{i,j,k=1}^{s} b_i a_{ij} a_{jk} c_k = \frac{1}{24}$ from (27), (29) and (30). And these are just fourth order consistency conditions obtained by Butcher [3] with the rooted trees. It seems that our algebraic criterion is more convenient to apply.

Remarkably, the internal vectors in Runge–Kutta methods should be expressed in form (7*) as a rule as in the above one–leg methods. But under (27), (29) and (31) etc., we find

$$w^{[\nu]} = w^{\nu}, \quad \nu=1, 2, \cdots, p.$$

So we can use just one vector w of (24).

5° Block implicit one–step methods. Starting from y_{nk} and $hf(y_{nk})$, we can compute $y_{nk+1}, y_{nk+2}, \cdots, y_{(n+1)k}$ and $hf(y_{nk}), hf(y_{nk+1}), \cdots, hf(y_{(n+1)k})$ simultaneously as follows:

$$\sum_{j=1}^{k} a_{ij}^* y_{nk+j} + a_{i0}^* y_{nk} = h \sum_{j=1}^{k} b_{ij}^* f(y_{nk+j}) + h b_{i0}^* f(y_{nk}), \quad i=1, 2, \cdots, k.$$

Set

$$U_n = (y_{nk+1}, y_{nk+2}, \cdots, y_{(n+1)k})^T,$$

$$\phi_n = (f(y_{nk+1}), f(y_{nk+2}), \cdots, f(y_{(n+1)k})^T;$$

$$A^* = (a_{ij}^*)_{i,j=1,2,\cdots,k}, \ a^* = (a_{10}^*, \cdots, a_{k0}^*)^T,$$

$$B^* = (b_{ij}^*)_{i,j=1,2,\cdots,k}, \ b^* = (b_{10}^*, \cdots, b_{k0}^*)^T.$$

Then we obtain the matrix–vector expression:

$$[A^*] U_n + a^* \otimes y_{nk} = h [B^*] \phi_n + h b^* \otimes f(y_{nk}).$$

Under usual assumption det $A^* = 0$, we have the standard form of block implicit one–step methods:

$$U_n+(A^{*-1}a^*)\otimes y_{nk}=h[A^{*-1}B^*]\phi_n+h(A^{*-1}b^*)\otimes f(y_{nk}).$$

According to the minimum requirements of consistency (i.e. $\rho_i(1)=0,\ i=1, 2, \cdots, k$), we find

$$-A^{*-1}a^*=e.$$

Set

$$A^{*-1}B^*=A,\quad A^{*-1}b^*=b.$$

Thus, we obtain a simplified form:

$$U_n=e\otimes y_{nk}+h[A]\phi_n+hb\otimes f(y_{nk}). \tag{32}$$

We take k internal vectors

$$Y^{(n)}=Y_1^{(n)}\oplus\cdots\oplus Y_k^{(n)}\sim y(x_{nk+1})\oplus\cdots\oplus y(x_{(n+1)k})$$

and 2 external vectors

$$y^{(n)}=y_{(n+1)k}\oplus hf(y_{(n+1)k}).$$

Bring the block implicit one-step methods (32) into the general linear method (3), and we have

$$C_{11}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k}\\ a_{21} & a_{22} & \cdots & a_{2k}\\ \vdots & & & \\ a_{k1} & a_{k2} & \cdots & a_{kk}\end{pmatrix},\quad C_{12}=\begin{pmatrix} 1 & b_1\\ 1 & b_2\\ \vdots & \vdots\\ 1 & b_k\end{pmatrix},$$

$$C_{21}=\begin{pmatrix} a_{k1} & a_{k2} & \cdots & a_{k,\,k-1} & a_{kk}\\ 0 & 0 & \cdots & 0 & 1\end{pmatrix},\quad C_{22}=\begin{pmatrix} 1 & b_k\\ 0 & 0\end{pmatrix}.$$

Now,

$$w=(1, 2, \cdots, k)^T;$$
$$\alpha=(1\ 0)^T,\ \beta=(0\ 1)^T,$$
$$\gamma=(k\ 0)^T,\ \delta=(0\ k)^T.$$

Remarkably, the shift between the external vectors $y^{(n)}$ and $y^{(n-1)}$ is in fact kh. Therefore the expressions $(\gamma-e)^\nu$ and $(\delta-e)^\nu$ should be replaced by $(\gamma-ke)^\nu$ and $(\delta-ke)^\nu$ respectively in the consistency condition (14).

Condition (13) is trivial. When $\nu=1$ from (14) we get

$$Ae+b=w.$$

In addition, the consistency of order $p>1$ requires

$$\nu Aw^{\nu-1}=w^\nu,\quad \nu=2, 3, \cdots, p.$$

It coincides with the conclusion of Shampine and Watts [4].

Taking into account the above two cases, we extend Theorem 1 as follows.

Theorem 2. *For the general linear methods* (3), *if the internal vectors*

$$Y_i^{(n)}\sim y(x_n)+hw_i^{[1]}y'(x_n)+\frac{h^2}{2}w_i^{[2]}y''(x_n)+\cdots$$
$$+\frac{h^p}{p!}w_i^{[p]}y^{(p)}(x_n)+\frac{h^{p+1}}{(p+1)!}w_i^{[p+1]}y^{(p+1)}(x_n)+o(h^{p+1}),\quad i=1, 2, \cdots, s \tag{7'}$$

and the external vectors

$$y_i^{(n-1)}\sim\zeta_i^{(n-1)}=\alpha_i y(x_0+(n-l)h+\gamma_i h)+\beta_i hy'(x_0+(n-l)h+\delta_i h),\quad i=1, 2, \cdots, k, \tag{10'}$$

then the necessary and sufficient conditions of consistency of order p are

$$C_{12}\alpha=e, \quad C_{22}\alpha=\alpha; \tag{13}$$

$$\begin{cases} C_{11}\left\{\frac{w^{[\nu-1]}}{(\nu-1)!}\right\}+C_{12}\left\{\alpha*\frac{(\gamma-le)^{\nu}}{\nu!}+\beta*\frac{(\delta-le)^{\nu-1}}{(\nu-1)!}\right\}=\frac{w^{[\nu]}}{\nu!}, \\ C_{21}\left\{\frac{w^{[\nu-1]}}{(\nu-1)!}\right\}+C_{22}\left\{\alpha*\frac{(\gamma-le)^{\nu}}{\nu!}+\beta*\frac{(\delta-le)^{\nu-1}}{(\nu-1)!}\right\}=\alpha*\frac{\gamma^{\nu}}{\nu!}+\beta*\frac{\delta^{\nu-1}}{(\nu-1)!}, \\ \nu=1, 2, \cdots, p. \end{cases} \tag{14'}$$

§ 5. On *B*-consistency

For the right-hand function f of the differential equation system, assume the following one-sided Lipschitz condition

$$\langle f(\bar{y})-f(y^*), \bar{y}-y^*\rangle\leqslant\mu\|\bar{y}-y^*\|^2. \tag{33}$$

The inequality holds in a certain neighborhood of the solution curve $y(x)$, i.e. it holds for $\forall\bar{y}, y^*\in O(y, r)$. The one-sided Lipschitz constant μ can be taken as the least upper bound of the logarithmic norm of the Jacobian $\frac{\partial f}{\partial y}$:

$$\mu=\sup_{O(y,r)}\mu\left(\frac{\partial f}{\partial y}\right). \tag{34}$$

Theorem 3. *For the general linear method (3), if there is only one internal vector $Y^{(n)}$ and $C_{11}\neq0$, then the consistency of order p implies the B-consistency of order p.*

Proof. With the symbol of (7)—(9), starting from

$$y^{(n-1)}=Z^{(n-1)}, \tag{35}$$

by (3) we have

$$\begin{cases} Y^{(n)}=C_{11}hf(Y^{(n)})+[C_{12}]Z^{(n-1)}, \\ y^{(n)}=[C_{21}]hf(Y^{(n)})+[C_{22}]Z^{(n-1)}. \end{cases} \tag{36}$$

According to (15), the consistency of order p means

$$\begin{cases} H^{(n)}=C_{11}hf(H^{(n)})+[C_{12}]Z^{(n-1)}+h^{p+1}L_{p+1}^{(1)}\otimes y^{(p+1)}(x_n)+o(h^{p+1}), \\ Z^{(n)}=[C_{21}]hf(H^{(n)})+[C_{22}]Z^{(n-1)}+h^{p+1}L_{p+1}^{(2)}\otimes y^{(p+1)}(x_n)+o(h^{p+1}), \end{cases} \tag{37}$$

where $L_{p+1}^{(1)}$ and $L_{p+1}^{(2)}$ denote the leading coefficient vectors of the local discretization error. Set

$$L_{p+1}=\max\{\|L_{p+1}^{(1)}\|, \|L_{p+1}^{(2)}\|\}.$$

Then,

$$\begin{cases} Y^{(n)}-H^{(n)}=C_{11}h\{f(Y^{(n)})-f(H^{(n)})\}-h^{p+1}L_{p+1}^{(1)}\otimes y^{(p)}(x_n)+o(h^{p+1}), & (38) \\ y^{(n)}-Z^{(n)}=[C_{21}]h\{f(Y^{(n)})-f(H^{(n)})\}-h^{p+1}L_{p+1}^{(2)}\otimes y^{(p)}(x_n)+o(h^{p+1}). & (39) \end{cases}$$

Taking the inner product on both sides of (38) with $Y^{(n)}-H^{(n)}$, by means of the one-sided Lipschitz condition (33), we obtain

$$\|Y^{(n)}-H^{(n)}\|^2\leqslant hc_{11}\mu\|Y^{(n)}-H^{(n)}\|^2+h^{p+1}L_{p+1}M_{p+1}\|Y^{(n)}-H^{(n)}\|,$$

where $\|y^{(p+1)}\|<M_{p+1}$. That is,

$$\|Y^{(n)}-H^{(n)}\| \leqslant \frac{1}{1-hC_{11}\mu} h^{p+1} L_{p+1} M_{p+1}. \tag{40}$$

Rewriting (38) as

$$h\{f(Y^{(n)})-f(H^{(n)})\}=\frac{1}{C_{11}}\{(Y^{(n)}-H^{(n)})+h^{p+1}L_{p+1}^{(1)}\otimes y^{(p+1)}(x_n)+o(h^{p+1})\},$$

and substituting into (39), we get

$$\|y^{(n)}-Z^{(n-1)}\| \leqslant \left(\frac{\|C_{21}\|}{|C_{11}|}\frac{2-hC_{11}\mu}{1-hC_{11}\mu}+1\right)L_{p+1}M_{p+1}h^{p+1}. \tag{41}$$

It shows that method (3) has the B-consistency of order p. This completes the proof.

Remark. What discussed by Frank, Schneid and Ueberhuber [2] is just the implicit Euler scheme, implicit midpoint rule and implicit trapezoidal rule. And the B-consistency of these three linear methods is included in the above Theorem 3.

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