ON A ONE-DIMENSIONAL DIFFERENCE SCHEME IN REACTION DIFFUSION*

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I. Introduction

Ludwig, Jones and Holling (1978) modelled the spruce budworm problem by using a scaled ordinary differential equation. Spatial effects were introduced by Ludwig, Aronson, and Weinberger (1979) by the addition of a diffusion term to the equation. Recently Guo Ben-yu et al. (1983) obtained some precise results for the bifurcation lengths in circular and rectangular regions. These analytic results are extended in the present paper to cover the case of difference equations in reaction diffusion. The analysis is restricted to one space dimension and only the linear and nonlinear logistic models are considered. Despite these restrictions, the techniques used and the comparison principles proved are useful for more general problems.

II. The Linear Model

In this section we consider the linear model of the spruce budworm problem. Let the region considered be an infinite strip of breadth l and $W(y, t)$ be the scaled density of the budworm population where (see Ludwig, Aronson and Weinberger, 1979)

$$\begin{cases} \frac{\partial W}{\partial t}-\frac{\partial^2 W}{\partial y^2}-W=0, & 0<y<l,\ t>0, \\ W(0, t)=W(l, t)=0, & t\geqslant 0, \\ W(y, 0)=W_0(y), & 0<y<l, \end{cases} \tag{1}$$

where $0\leqslant U_0(x)\leqslant M_0$. Let $y=lx$, $U(x, t)=W\left(\frac{y}{l}, t\right)$ and $\varepsilon=\frac{1}{l^2}$. Then (1) becomes

$$\begin{cases} \frac{\partial U}{\partial t}-\varepsilon\frac{\partial^2 U}{\partial x^2}-U=0, & 0<x<1,\ t>0, \\ U(0, t)=U(1, t)=0, & t\geqslant 0, \\ U(x, 0)=U_0(x), & 0<x<1. \end{cases} \tag{2}$$

We cover the region $[0\leqslant x\leqslant 1]\times[t\geqslant 0]$ by a rectangular grid, where h and τ are the mesh sizes of the variables x and t respectively; also, $Nh=1$ where N is an integer. We define

* Received July 28, 1984. Revised July 7, 1986.

$$I_h=\{x\mid x=h,\ 2h,\ \cdots,\ (N-1)h\},\quad \bar{I}_h=I_h+\{0\}+\{1\}.$$

Let $\eta^k(x)$ be the value of the mesh function η at point $x\in\bar{I}_h$, $t=k\tau$ $(k\geqslant 0)$. We use the following notations

$$\eta_x^k(x)=\frac{1}{h}[\eta^k(x+h)-\eta^k(x)],\qquad \eta_{\bar x}^k(x)=\eta_x^k(x-h),$$

$$\eta_{x\bar x}^k(x)=\frac{1}{h^2}[\eta^k(x+h)-2\eta^k(x)+\eta^k(x-h)],$$

and

$$\eta_t^k(x)=\frac{1}{\tau}[\eta^{k+1}(x)-\eta^k(x)].$$

We introduce the discrete scalar product and the norms as follows:

$$(\eta^k,\ \xi^k)=h\sum_{x\in I_h}\eta^k(x)\cdot\xi^k(x),$$

$$\|\eta^k\|^2=(\eta^k,\ \eta^k),\qquad |\eta^k|_1^2=\frac{1}{2}\|\eta_x^k\|^2+\frac{1}{2}\|\eta_{\bar x}^k\|^2,\qquad \|\eta^k\|_\infty=\max_{x\in I_h}|\eta^k(x)|.$$

It is clear that

$$-(\eta^k,\ \eta_{x\bar x}^k)=|\eta^k|_1^2+\frac{1}{2h}[\eta(h)]^2+\frac{1}{2h}[\eta(1-h)]^2. \tag{3}$$

Let $u^k(x)$ be the approximation to $U^k(x)$. The Crank-Nicolson scheme for solving (2) is

$$\begin{cases} u_t^k(x)-\frac{\varepsilon}{2}(u_{x\bar x}^k(x)+u_{x\bar x}^{k+1}(x))-\frac{1}{2}[u^k(x)+u^{k+1}(x)]=0, & x\in I_h,\ k\geqslant 0,\\ u^k(0)=u^k(1)=0, & k\geqslant 0,\\ u^0(x)=U_0(x), & x\in I_h. \end{cases} \tag{4}$$

Let

$$u^k(x)=\sum_{\beta=1}^{N-1}a_\beta b^k(\beta)\sin\beta\pi x,\quad x\in\bar{I}_h,\ k\geqslant 0,$$

where

$$U_0(x)=\sum_{\beta=1}^{N-1}a_\beta\sin\beta\pi x,\quad x\in\bar{I}_h.$$

Then

$$b(\beta)=\frac{1-\frac{2\varepsilon\tau}{h^2}\sin^2\frac{\beta\pi h}{2}+\frac{\tau}{2}}{1+\frac{2\varepsilon\tau}{h^2}\sin^2\frac{\beta\pi h}{2}-\frac{\tau}{2}}.$$

We define

$$\varepsilon_h^*=\frac{h^2}{4\sin^2\frac{\pi h}{2}}$$

and let $w^k(x)$ be the solution of (4) with $w^0(x)\equiv M_0$. Then

$$w^k(x)=\frac{2M_0}{N}\sum_{\beta=1}^{\frac{N}{2}}\frac{b^k(\beta)\cos\frac{\beta\pi}{2N}\sin(2\beta-1)\pi x}{\sin\frac{\beta\pi}{2N}}.$$

Clearly $w^k(x)\leqslant w^k\left(\frac{1}{2}\right)$ and $w^k\left(\frac{1}{2}\right)\to 0$ as $k\to\infty$ provided $\varepsilon>\varepsilon_h^*$.

Now suppose

$$\tau\leqslant\min\left(2,\ \frac{2h^2}{2\varepsilon-h^2}\right).$$

By Lemma A_2 (see Appendix), $0\leqslant u^k(x)\leqslant w^k(x)$ and thus $u^k(x)\to 0$ as $k\to\infty$ as long as $\varepsilon > \varepsilon_h^*$. On the other hand, if $\varepsilon < \varepsilon_h^*$ and $u^0(x) = \delta \sin \pi x$, then for all $\delta > 0$ and $x \in I_h$,

$$u^k(x) = b^k(1) u^0(x) \to \infty \quad \text{as } k \to \infty.$$

Now let

$$l_h^* = \frac{1}{\sqrt{\varepsilon_h^*}} = \frac{2 \sin \frac{\pi h}{2}}{h}$$

and we conclude that

(i) if $l < l_h^*$, then for any initial value $u^0(x)$ and $x \in \bar{I}_h$, $u^k(x) \to 0$ as $k \to \infty$;

(ii) if $l > l_h^*$, then there are solutions with arbitrarily small initial values which can grow without any bound for all $x \in I_h$ as $k \to \infty$.

The value l_h^* is the critical size of budworm refuge for the discrete model (4). As is well known, the critical size of the original problem (1) for the budworm refuge is $l^* = \pi$ (see Ludwig, Aronson and Weinberger, 1979) and so $l_h^* \to l^*$ as $h \to 0$.

III. The Logistic Model and Its Steady Solution

We consider the logistic model (see Ludwig, Aronson and Weinberger, 1979) which is given by

$$\begin{cases} \frac{\partial W}{\partial t} - \frac{\partial^2 W}{\partial y^2} - W(1-W) = 0, & 0<y<l,\ t>0, \\ W(0, t) = W(l, t) = 0, & t \geq 0, \\ W(y, 0) = W_0(y), & 0<y<l, \end{cases} \tag{5}$$

or

$$\begin{cases} \frac{\partial U}{\partial t} - \varepsilon \frac{\partial^2 U}{\partial x^2} - U(1-U) = 0, & 0<x<1,\ t>0, \\ U(0, t) = U(1, t) = 0, & t \geq 0, \\ U(x, 0) = U_0(x), & 0<x<1. \end{cases} \tag{6}$$

The corresponding steady problem is

$$\begin{cases} \varepsilon \frac{\partial^2 V}{\partial x^2} + V(1-V) = 0, & 0<x<1, \\ V(0) = V(1) = 0. \end{cases} \tag{7}$$

The Crank-Nicolson type scheme for solving (6) is

$$\begin{cases} u_t^k(x) - \frac{\varepsilon}{2}(u_{x\bar{x}}^k(x) + u_{x\bar{x}}^{k+1}(x)) - \frac{1}{2}(u^k(x) + u^{k+1}(x)) + (u^k(x))^2 = 0, & x \in I_h,\ k \geq 0, \\ u^k(0) = u^k(1) = 0, & k \geq 0, \\ u^0(x) = U_0(x), & x \in I_h, \end{cases} \tag{8}$$

and the corresponding steady problem is

$$\begin{cases} \varepsilon v_{x\bar{x}}(x) + v(x)(1 - v(x)) = 0, & x \in I_h, \\ v(0) = v(1) = 0. \end{cases} \tag{9}$$

Now we look for the condition for (9) to have a positive solution. We take the discrete scalar product of (9) with $v(x)$. From (3), it follows that for a positive solution,

$$\varepsilon|v|_1^2+\frac{\varepsilon}{2h}v^2(h)+\frac{\varepsilon}{2h}v^2(1-h)-\|v\|^2\leqslant 0.$$

Let

$$m_h=\sup_{\eta\neq 0}\frac{\|\eta\|^2}{|\eta|_1^2+\frac{1}{2h}\eta^2(h)+\frac{1}{2h}\eta^2(1-h)}.$$

Then

$$(\varepsilon-m_h)\left[|v|_1^2+\frac{1}{2h}v^2(h)+\frac{1}{2h}v^2(1-h)\right]\leqslant 0.$$

If $m_h<\varepsilon$, then $|v|_1^2=0$ and so $v(x)\equiv 0$ for $x\in\bar{I}_h$. Indeed $\frac{1}{m_h}$ is the smallest eigenvalue of

$$\begin{cases}\phi_{x\bar{x}}(x)+\lambda_h\phi(x)=0, & x\in I_h,\\ \phi(0)=\phi(1)=0.\end{cases}$$

We have

$$\lambda_h=\frac{4}{h^2}\sin^2\frac{\beta\pi h}{2},\quad 1\leqslant\beta\leqslant N-1,$$

and so $m_h=\varepsilon_h^*$. Thus if $\varepsilon>\varepsilon_h^*$, then (9) has no positive solution.

We shall show that ε_h^* is the first bifurcation point of (9). First we introduce the discrete Green function $g_h(x, x')$, given by

$$\begin{cases}-g_{h,x\bar{x}}(x, x')=\frac{1}{h}\delta(x, x'), & x\in I_h,\ x'\in\bar{I}_h,\\ g_h(0, x')=g_h(1, x')=0, & x'\in\bar{I}_h.\end{cases}$$

We next define the operator L_h as

$$(L_h\eta)(x)=h\sum_{x'\in I_h}g_h(x', x)\eta(x')[1-\eta(x')]$$

and so (9) is now equivalent to the operator equation

$$\varepsilon v=L_h v. \tag{10}$$

Let B_h be the discrete function space with the norm $\|\eta\|_{B_h}=\|\eta\|_\infty$ and for all $\eta\in B_h$, $\eta(0)=\eta(1)=0$. Let θ be the null element in B_h. Then $L_h(\theta)=\theta$. Let $L_h'(\eta)$ and $L_h''(\eta)$ be the first and second order Frechet derivatives of L_h respectively. We can then prove that on some open neighbourhood of θ,

$$\|L_h''(\eta)\|\leqslant M_1<\infty, \tag{11}$$

where M_1 is a positive constant independent of h and $\|\cdot\|$ denotes the norm of the operator $L_h''(\eta)$.

We now consider the following eigenvalue problem corresponding to (10)

$$\varepsilon_h\phi_h(x)=L_h'(\theta)\phi_h(x)$$

which is equivalent to

$$\begin{cases}\varepsilon_h\phi_{h,x\bar{x}}(x)+\phi_h(x)=0, & x\in I_h,\\ \phi_h(0)=\phi_h(1)=0.\end{cases}$$

Let $\sigma_h(\theta)$ be the spectrum of the operator $L_h'(\theta)$. Then

$$\sigma_h(\theta)=\left\{\frac{h^2}{4\sin^2\frac{\beta\pi h}{2}}\Big/1\leqslant\beta\leqslant N-1\right\}.$$

The largest eigenvalue is ε_h^* with the corresponding eigenfunction $\phi_h^*(x)=\sin\pi x$.

Now let $\sigma_h^*=\sigma_h(\theta)-\{\varepsilon_h^*\}$ and $H(\varepsilon_h^*)$ and $H(\sigma_h^*)$ denote the null space and the range of $\varepsilon_h^* - L_h'(\theta)$ respectively. Then

$$B_h = H(\varepsilon_h^*) \oplus H(\sigma_h^*). \tag{12}$$

E_h and F_h denote the projection operators of B_h onto $H(\varepsilon_h^*)$ and $H(\sigma_h^*)$ respectively. We define the operator A_h as

$$A_h = \frac{1}{2\pi i} \int_{\Gamma_h} \frac{1}{\varepsilon_h^* - z} [z - L_h'(\theta)]^{-1} dz,$$

where the curve Γ_h is composed of a finite number of rectifiable curves. The curve Γ_h contains σ_h^* in its interior and ε_h^* is in the exterior to Γ_h. It can now be proved that

$$\|A_h\| \leqslant M_2,$$

where M_2 is independent of h and

$$A_h[\varepsilon_h^* - L_h'(\theta)] = [\varepsilon_h^* - L_h'(\theta)] A_h = F_h. \tag{13}$$

Returning to equation (10) we let $\varepsilon = \varepsilon_h^* + \delta_h$ and expand L_h about θ to obtain

$$(\varepsilon_h^* - L_h'(\theta)) v = -\delta_h v + R_h(v), \tag{14}$$

where

$$R_h(v) = L_h(v) - L_h(\theta) - L_h'(\theta) v = L_h(v) - L_h'(\theta) v.$$

From (11) it follows that for all v sufficiently small

$$\|R_h(v)\|_{B_h} \leqslant \frac{M_1}{2} \|v\|_{B_h}^2. \tag{15}$$

Now let c be a parameter where $c \neq 0$, and let $v_c = c[\phi_h^* + w_h]$ where $w_h \in H(\sigma_h^*)$. By using (12) and (13), (14) is equivalent to the system

$$w_h = -\delta_h A_h w_h + \frac{1}{c} A_h R_h(c[\phi_h^* + w_h]), \tag{16}$$

$$\delta_h \phi_h^* = \frac{1}{c} E_h R_h(c[\phi_h^* + w_h]) = M_h(c, \delta_h) \phi_h^*. \tag{17}$$

Now assume

$$|\delta_h| \leqslant \delta^0 < \frac{1}{\sup\limits_{h<h_0} \|A_h\|},$$

and so (16) becomes

$$w_h = \frac{1}{c} (1 + \delta_h A_h)^{-1} A_h R_h(c[\phi_h^* + w_h]) = N_h(w_h; c, \delta_h). \tag{18}$$

We shall show that (18) has a solution $w_h(c, \delta_h)$ by using the contractive mapping theorem. Using (15), we obtain

$$\|N_h(w_h; c, \delta_h)\|_{B_h} \leqslant c d_1 \tag{19}$$

for all sufficiently small c, δ_h and w_h with an appropriate $d_1 > 0$. For $\|w_h\|_{B_h} \leqslant r \leqslant r^0$ where r^0 is suitably small, choose $|c| d_1 \leqslant r$, and then $N_h(\cdot; c, \delta_h)$ maps $S_h(\theta, r)$ into $S_h(\theta, r)$ where

$$S_h(\theta, r) = \{w_h / \|w_h\|_{B_h} \leqslant r\}.$$

Now we show that N_h is contractive. From (14) and (11) it follows that for all sufficiently small $v^{(1)}$ and $v^{(2)}$,

$$\|R_h(v^{(1)}) - R_h(v^{(2)})\|_{B_h} \leqslant M_1 [\|v^{(2)}\|_{B_h} + \frac{1}{2} \|v^{(1)} - v^{(2)}\|_{B_h}] \|v^{(1)} - v^{(2)}\|_{B_h}.$$

Applying this to N_h, we obtain

$$\|N_h(v^{(1)}; c, \delta_h) - N_h(v^{(2)}; c, \delta_h)\|_{B_h} \leqslant d_2 |c| \|v^{(1)} - v^{(2)}\|_{B_h} \tag{20}$$

for all sufficiently small c, δ_h, $v^{(1)}$ and $v^{(2)}$, with $d_2 > 0$. Thus if $|c| \leqslant c^0 < 1/d_2$, then

N_h is a contractive operator. Combining (19) with (20) on some $S(\theta, r)$ and using the contractive mapping theorem, there is a unique solution $w_h(c, \delta_h)$ in $S_h(\theta, r)$ and $w_h=N_h(w_h; c, \delta_h)$. Moreover from (19),

$$|w_h(c, \delta_h)| \leqslant d_1|c| = O(c) \tag{21}$$

uniformly for all small values of δ_h and h. It means that $w_h(c, \delta_h)$ is a uniformly continuous function of c and δ_h.

As for (17), we use the same technique. For all small c and δ_h, by using (15) and (21), we obtain

$$|M_h(c, \delta_h)| \leqslant d_3|c|$$

for $d_3>0$ and all small c. Let $|\delta_h| \leqslant \nu$ and choose $|cd_3| \leqslant \nu$. This implies that $M_h(c, \cdot)$ maps the interval $[-\nu, \nu]$ into $[-\nu, \nu]$. We have from (15), (11) and (21) that

$$\begin{aligned}
&|M_h(c, \delta_h^{(1)}) - M_h(c, \delta_h^{(2)})| \\
&\leqslant \frac{1}{|c|}\|E_h\| \, \|R_h(c[\phi_h^*+w_h(c, \delta_h^{(1)})]) - R_h(c[\phi_h^*+w_h(c, \delta_h^{(2)})])\|_{B_h} \\
&\leqslant cd_4\|w_h(c, \delta_h^{(1)}) - w_h(c, \delta_h^{(2)})\|_{B_h} \leqslant c^2 d_5 |\delta_h^{(1)} - \delta_h^{(2)}|
\end{aligned}$$

for appropriate constants d_4, $d_5>0$. Now applying the contractive mapping theorem to the equation $\delta_h^* = M_h(c, \delta_h)$, we obtain the existence of $\delta_h(c)$, $|c| \leqslant c^0$ for some $c^0>0$ and it follows easily that $\delta_h(c)$ depends continuously on c and $\delta_h=0$ for $c=0$.

The previous statements show that problem (9) has a unique solution $v_0(x)$ and $\varepsilon=\varepsilon_h^*+\delta_h$, for which

$$\begin{aligned}
&v_0 = c[\phi_h^*+w_h], \\
&w_h \in \text{Range}(\varepsilon_h^* - L_h'(\theta)), \\
&w_h = O(c), \quad \delta_h(c) = O(c).
\end{aligned}$$

The previous analysis is similar to that in Atkinson (1977). We take c to be sufficiently small and positive. Since $\phi_h^*>0$ for $0<x<1$, so $v_0>0$. As is shown before, if $\varepsilon>\varepsilon_h^*$, then (9) has only the zero solution and so (9) has a unique positive solution $v_0(x)$ only for $\varepsilon<\varepsilon_h^*$.

Next we shall show that (9) has a unique positive solution for all $\varepsilon<\varepsilon_h^*$. Let $\varepsilon<\varepsilon'<\varepsilon_h^*$ with $|\varepsilon'-\varepsilon_h^*|$ sufficiently small. Then there is a unique positive solution $\varphi(x)$ of the following problem

$$\begin{cases} -\varepsilon'\varphi_{x\bar{x}}(x) - \varphi(x)(1-\varphi(x)) = 0, & x \in I_h, \\ \varphi(x) = 0, & x = 0, 1, \end{cases}$$

from which and $0<\varphi(x)<1$ for all $x \in I_h$, we have

$$-\varepsilon\varphi_{x\bar{x}}(x) - \varphi(x)(1-\varphi(x)) = (\varepsilon'-\varepsilon)\varphi_{x\bar{x}}(x) = \frac{\varepsilon'-\varepsilon}{\varepsilon'}\varphi(x)(\varphi(x)-1) < 0.$$

Thus $\varphi(x)$ is a strict subsolution of (9) (see Appendix). Obviously $\psi(x)=1+\beta$ $(\beta>0)$ is a strict supersolution of (9). By Lemma A_1 (see Appendix), (9) has at least one positive solution.

Finally we consider the uniqueness of the positive solution. Assume $v^{(1)}(x)$ and $v^{(2)}(x)$ are two such solutions. Suppose $v^{(2)}(x)>v^{(1)}(x)$ for $x \in E_h^* \subseteq I_h$. Choose $\beta>1$ such that

$$v^{(2)}(x) < \beta v^{(1)}(x), \quad x \in I_h$$

and that

$$v^{(2)}(x^{(0)})\geqslant\beta v^{(1)}(x^{(0)}),\quad x^{(0)}\in I_h.$$

Let $w(x)=\beta v^{(1)}(x)$. Then

$$-\varepsilon w_{x\bar{x}}(x)-w(x)(1-w(x))=\beta(\beta-1)[v^{(1)}(x)]^2>0$$

and so $w(x)$ is a strict supersolution of (9). Let $\xi^k(x)$ be the solution of (8) with $\xi^0(x)=w(x)$ and

$$\tau<\min\left(2,\frac{2h^2}{2\varepsilon+4\beta h^2-h^2}\right).$$

By Lemma A_4 (see Appendix), $\xi^k(x)$ is a strictly decreasing function of k for all $x\in I_h$ and so $\xi^k(x)<w(x)$. On the other hand, Lemma A_2 leads to

$$v^{(2)}(x)\leqslant\xi^k(x),\quad x\in\bar{I}_h,\ k\geqslant0,$$

and thus

$$v^{(2)}(x^{(0)})\leqslant\xi^k(x^{(0)})<w(x^{(0)})\leqslant\beta v^{(1)}(x^{(0)}),$$

which is contrary to the assumption.

Now we conclude that l_h^* is the critical size of (9), that is

(i) if $l<l_h^*$, then (9) has only the zero solution,

(ii) if $l>l_h^*$, then (9) has a unique positive solution.

As is well known, the critical size in the original problem (7) is $l^*=\pi$. Clearly $l_h^*\to l^*$ as $h\to0$.

IV. The Asymptotic Behaviour of the Logistic Model

In this section we consider the asymptotic behaviour of the solution of (8), denoted by $u^k(x)$. Suppose $0\leqslant u^0(x)\leqslant M_0$ and $M_3=\max(1, M_0)$. If

$$\tau\leqslant\tau^*\leqslant\min\left(2,\frac{2h^2}{2\varepsilon+4M_3h^2-h^2}\right),$$

then by Lemma A_3, $0\leqslant u^k(x)\leqslant M_3$.

Now let $u^k(x)$ and $w^k(x)$ be the solution of (8) and (4) respectively and $u^0(x)=w^0(x)\geqslant0$. Then

$$\begin{cases}u_t^k(x)-\dfrac{\varepsilon}{2}(u_{x\bar{x}}^{k+1}(x)+u_{x\bar{x}}^k(x))-\dfrac{1}{2}(u^{k+1}(x)+u^k(x))=-[u^k(x)]^2\leqslant0, & x\in I_h,\ k\geqslant0,\\ u^k(x)=0, \qquad x=0,\ 1,\ k\geqslant0,\\ u^0(x)=w^0(x),\quad x\in\bar{I}_h.\end{cases}$$

By using Lemma A_2 and Lemma A_3, we have $0\leqslant u^k(x)\leqslant w^k(x)$, and thus $u^k(x)\to0$ as $k\to\infty$ provided $\varepsilon>\varepsilon_h^*$.

Now assume $\varepsilon<\varepsilon_h^*$. Then (9) has a unique positive solution $v(x)$. For simplicity, assume $0<m\leqslant u^0(x)\leqslant M_0$ for all $x\in I_h$. Let $\varepsilon'<\varepsilon_h^*$ and $|\varepsilon'-\varepsilon_h^*|$ be sufficiently small. Let $\varphi(x)$ denote the solution of (9) with $\varepsilon=\varepsilon'$. Then $0<\|\varphi\|_\infty<m$. Let $\xi^k(x)$ and $\eta^k(x)$ be the solution of the following problems

$$\begin{cases}\xi_t^k(x)-\dfrac{\varepsilon}{2}(\xi_{x\bar{x}}^k(x)+\xi_{x\bar{x}}^{k+1}(x))-\dfrac{1}{2}(\xi^k(x)+\xi^{k+1}(x))+[\xi^k(x)]^2=0, & x\in I_h,\ k\geqslant0,\\ \xi^k(x)=0,\qquad x=0,\ 1,\ k\geqslant0,\\ \xi^0(x)=M_3,\quad x\in\bar{I}_h,\end{cases}$$

and

$$\begin{cases} \eta_t^k(x)-\frac{\varepsilon}{2}(\eta_{x\bar{x}}^k(x)+\eta_{x\bar{x}}^{k+1}(x))-\frac{1}{2}(\eta^k(x)+\eta^{k+1}(x))+[\eta^k(x)]^2=0, & x\in I_h,\ k\geqslant 0, \\ \eta^k(x)=0, & x=0,\ 1,\ k\geqslant 0, \\ \eta^0(x)=\varphi(x), & x\in \bar{I}_h. \end{cases}$$

Because $\xi^0(x)$ and $\eta^0(x)$ are supersolution and subsolution of (9) respectively, Lemma A_4 and Lemma A_5 lead to

$$v(x)=\lim_{k\to\infty}\xi^k(x)=\lim_{k\to\infty}\eta^k(x).$$

By applying Lemma A_2, we have

$$\eta^k(x)\leqslant u^k(x)\leqslant \xi^k(x)$$

and thus $u^k(x)\to v(x)$ as $k\to\infty$.

The conclusion is as follows:

(i) If $l<l_h^*$, then for any $u^0(x)\geqslant 0$ and all $x\in\bar{I}_h$, $u^k(x)\to 0$ as $k\to\infty$.

(ii) If $l>l_h^*$, $u^0(x)\geqslant 0$ and $u^0(x)\not\equiv 0$, then $u^k(x)\to v(x)$ as $k\to\infty$.

As is known, if $l<l^*=\pi$, then $U(x,t)\to 0$; if $l>l^*$, $U_0(x)\geqslant 0$ and $U_0(x)\not\equiv 0$, then $U(x,t)\to V(x)$ as $t\to\infty$ where $V(x)$ is the unique positive solution of (7). Since $l_h^*\to l^*$ as $h\to 0$, the asymptotic solution of the discretised problem (8) tends to that of (6) as $h\to 0$.

V. The Convergence of the Approximate Solution

Sometimes we want to know not only the asymptotic but also the temporal behaviour. Thus we must consider the accuracy of the approximate solution at each value of the time t. Let $\tilde{u}^k(x)=u^k(x)-U^k(x)$. Then

$$\begin{cases} \tilde{u}_t^k(x)-\frac{\varepsilon}{2}(\tilde{u}_{x\bar{x}}^k(x)+\tilde{u}_{x\bar{x}}^{k+1}(x)) \\ \quad -\frac{1}{2}(\tilde{u}^k(x)+\tilde{u}^{k+1}(x))+[u^k(x)+U^k(x)]\tilde{u}^k(x)=R^k(x), & x\in I_h,\ k\geqslant 0, \\ \tilde{u}^k(x)=0, & x=0,\ 1,\ k\geqslant 0, \\ \tilde{u}^0(x)=0, & x\in I_h, \end{cases} \tag{22}$$

where $R^k(x)$ is the truncation error. If $U(x,t)$ is smooth enough, then

$$|R^k(x)|\leqslant M_4(\tau+h^2).$$

We also suppose

$$0\leqslant u^k(x)+U^k(x)\leqslant M_5=2M_3.$$

From (22) it follows that

$$\begin{aligned}&\left(1+\frac{\varepsilon\tau}{h^2}-\frac{\tau}{2}\right)\tilde{u}^{k+1}(x)-\frac{\varepsilon\tau}{2h^2}(\tilde{u}^{k+1}(x-h)+\tilde{u}^{k+1}(x+h)) \\ &=\left(1-\frac{\varepsilon\tau}{h^2}+\frac{\tau}{2}-\tau u^k(x)-\tau U^k(x)\right)\tilde{u}^k(x)+\frac{\varepsilon\tau}{2h^2}(\tilde{u}^k(x-h) \\ &\quad+\tilde{u}^k(x+h))+\tau R^k(x).\end{aligned}$$

If

$$\tau\leqslant\min\left(2,\frac{2h^2}{2\varepsilon+4M_5h^2-h^2}\right),$$

then from the maximum principle we obtain

$$\left(1-\frac{\tau}{2}\right)\|\tilde{u}^{k+1}\|_\infty \leqslant \left(1+\frac{\tau}{2}+M_6\tau\right)\|\tilde{u}^k\|_\infty+\tau\|R^k\|_\infty$$

and thus

$$\|\tilde{u}^k\|_\infty \leqslant M_7(1+M_8\tau)^k(\tau+h^2),$$

which implies the convergence of $u^k(x)$ to $U^k(x)$ uniformly for all $k\tau \leqslant t \leqslant T$, T being any fixed positive constant.

Appendix

Let $f(z)$ be a continuous function. We consider the following steady problem

$$\begin{cases} -\varepsilon v_{x\bar{x}}(x)-f(v(x))=0, & x\in I_h, \\ v(x)=0, & x=0,1. \end{cases} \tag{A$_1$}$$

Define the discrete Green function as follows:

$$\begin{cases} -g_{h,x\bar{x}}(x, x')=\frac{1}{h}\delta(x, x'), & x\in I_h,\ x'\in \bar{I}_h, \\ g_h(x, x')=0, & x=0,1,\ x'\in \bar{I}_h, \end{cases}$$

and $F_h=\frac{1}{\varepsilon}G_h\circ f$ where

$$G_h\eta(x)=h\sum_{x'\in I_h} g_h(x', x)\eta(x').$$

Then (A_1) is identical to the operator equation $v=F_hv$.

Definition A$_1$. *If*

$$\begin{cases} -\varepsilon\eta_{x\bar{x}}(x)-f(\eta(x))\geqslant 0, & x\in I_h, \\ \eta(x)\geqslant 0, & x=0,1, \end{cases}$$

then we say that $\eta(x)$ is a supersolution of (A_1). In particular, if one of the above inequalities holds strictly, then we say that $\eta(x)$ is a strict supersolution of (A_1).

Definition A$_2$. *If*

$$\begin{cases} -\varepsilon\eta_{x\bar{x}}(x)-f(\eta(x))\leqslant 0, & x\in I_h, \\ \eta(x)\leqslant 0, & x=0,1, \end{cases}$$

then we say that $\eta(x)$ is a subsolution of (A_1). In particular, if one of the above inequalities holds strictly, then we say that $\eta(x)$ is a strict subsolution of (A_1).

Now let $\varphi(x)$ and $\psi(x)$ be two continuous functions such that $\varphi(x)\leqslant\psi(x)$ for all $x\in\bar{I}_h$. We define

$$f(\varphi,\psi,\sigma)(z)=\begin{cases} \sigma f(\psi(x)), & \text{for } z>\psi(x), \\ \sigma f(z), & \text{for } \varphi(x)\leqslant z\leqslant\psi(x), \\ \sigma f(\varphi(x)), & \text{for } z<\varphi(x), \end{cases}$$

and

$$F_h(\varphi,\psi,\sigma)=\frac{1}{\varepsilon}G_h\circ f(\varphi,\psi,\sigma).$$

Clearly $F_h(\varphi,\psi,\sigma)$ is a continuous operator.

Let

$$K(\varphi,\psi)=\{w(x)/\varphi(x)\leqslant w(x)\leqslant\psi(x), \text{ for all } x\in\bar{I}_h\},$$

the interior of which is denoted by $\mathring{K}(\varphi,\psi)$. It is easy to show that the fixed points of $F_h(\varphi,\psi,1)$ in $K(\varphi,\psi)$ are the solutions of (A_1) in $K(\varphi,\psi)$.

Lemma A$_1$. *If $\varphi(x)$ and $\psi(x)$ are strict subsolution and supersolution of* (A$_1$) *respectively, $\varphi(x)\leqslant\psi(x)$, then F_h has at least one fixed point in $\dot{K}(\varphi, \psi)$.*

Proof. We first prove that $F_h=F_h(\varphi, \psi, 1)$ in $K(\varphi, \psi)$. In fact for all $v(x)\in K(\varphi, \psi)$ and $x\in\bar{I}_h$, we have $\varphi(x)\leqslant v(x)\leqslant\psi(x)$ and thus

$$F_h(\varphi, \psi, 1)v=\frac{1}{\varepsilon} G_h\circ f(\varphi, \psi, 1)(v)=\frac{1}{\varepsilon} G_h\circ f(v)=F_h v.$$

We shall next show that all fixed points of $F_h(\varphi, \psi, 1)$ are in $\dot{K}(\varphi, \psi)$. To see this, we assume that $v(x)$ is one of the fixed points of $F_h(\varphi, \psi, 1)$ and let

$$E_h^-=\{x\in\bar{I}_h/v(x)<\varphi(x)\}, \qquad E_h^+=\{x\in\bar{I}_h/v(x)>\psi(x)\}.$$

In general E_h^- and E_h^+ are composed of a finite number of connected sets. Without losing any generality we suppose that E_h^- is a connected set as well as E_h^+. We have

$$f(\varphi, \psi, 1)(v(x))=f(\varphi(x))\geqslant-\varepsilon\varphi_{x\bar{x}}(x), \quad \text{for } x\in\bar{E}_h.$$

Because $v=F_h(\varphi, \psi, 1)v$, we have

$$f(\varphi, \psi, 1)(v(x))=-\varepsilon v_{x\bar{x}}(x)$$

and so

$$-\varepsilon\varphi_{x\bar{x}}(x)\leqslant-\varepsilon v_{x\bar{x}}(x).$$

On the other hand we have $\varphi(x)\leqslant v(x)$ for all $x\in\partial E_h^-$ and so

$$\varphi(x)\leqslant v(x) \quad \text{for } x\in\bar{E}_h^-,$$

which is contrary to the definition of E_h^-. Thus E_h^- is empty. Similarly, E_h^+ is empty. Therefore $v\in K(\varphi, \psi)$. Furthermore, we can show that $v\in\dot{K}(\varphi, \psi)$.

We now prove that there exists a sufficiently large positive constant, denoted by r, such that all of the fixed points of $F_h(\varphi, \psi, \sigma)$ are in B_r where

$$B_r=\{w/\|w\|_\infty<r\}.$$

Indeed, if v is a fixed point of $F_h(\varphi, \psi, \sigma)$, then

$$\|v\|_\infty=\|F_h(\varphi, \psi, \sigma)v\|_\infty\leqslant\frac{C_1}{\varepsilon}\|f(\varphi, \psi, \sigma)(v)\|_\infty.$$

Because

$$\begin{aligned}\|f(\varphi, \psi, \sigma)(v)\|_\infty&=\max_{x\in\bar{I}_h}|f(\varphi, \psi, \sigma)(v(x))|\\&\leqslant\max\{\max_{x\in\bar{I}_h}|\sigma f(\varphi(x))|, \max_{x\in\bar{I}_h}|\sigma f(\psi(x))|, \max_{\substack{\varphi(x)\leqslant y\leqslant\psi(x)\\ x\in\bar{I}_h}}|\sigma f(y)|\},\end{aligned}$$

so there exists a constant $r>0$ independent of v, such that $\|v\|_\infty<r$.

Finally we obtain

$$\begin{aligned}\deg(1-F_h(\varphi, \psi, 1), B_r, 0)&=\deg(1-F_h(\varphi, \psi, 0), B_r, 0)\\&=\deg(1, B_r, 0)=1.\end{aligned}$$

Hence $F_h(\varphi, \psi, 1)$ has at least one fixed point in $\dot{K}(\varphi, \psi)$.

Combining the above statements, we complete the proof.

Now we consider the unsteady problem. Define

$$\begin{aligned}D(h, \tau, \varepsilon, a)\eta^k(x)=\eta_t^k(x)&-\frac{\varepsilon}{2}(\eta_{x\bar{x}}^k(x)+\eta_{x\bar{x}}^{k+1}(x))\\&-\frac{1}{2}(\eta^k(x)+\eta^{k+1}(x))+a(\eta^k(x))^2, \quad a>0.\end{aligned}$$

Lemma A₂. *Assume*

$$\max_{x,k} \xi^k(x) \leqslant C_2, \quad \tau \leqslant \min\left(2, \frac{2h^2}{2\varepsilon + 4aC_2h^2 - h^2}\right),$$

and

$$\begin{cases} D(h, \tau, \varepsilon, a)\eta^k(x) \leqslant D(h, \tau, \varepsilon, a)\xi^k(x), & x \in I_h, \ k \geqslant 0, \\ \eta^k(x) \leqslant \xi^k(x), & x = 0, 1, \ k \geqslant 0, \\ \eta^0(x) \leqslant \xi^0(x), & x \in \bar{I}_h. \end{cases}$$

Then for all $x \in \bar{I}_h$ and $k \geqslant 0$,

$$\eta^k(x) \leqslant \xi^k(x).$$

Proof. Put $\eta^k(x) = \xi^k(x) + \tilde{\eta}^k(x)$. We obtain

$$\tilde{\eta}^k_t(x) - \frac{\varepsilon}{2}(\tilde{\eta}^k_{x\bar{x}}(x) + \tilde{\eta}^{k+1}_{x\bar{x}}(x)) - \frac{1}{2}(\tilde{\eta}^k(x) + \tilde{\eta}^{k+1}(x))$$
$$+ a[\tilde{\eta}^k(x)]^2 + 2a\tilde{\eta}^k(x)\xi^k(x) \leqslant 0$$

which leads to

$$\left(1 + \frac{\varepsilon\tau}{h^2} - \frac{\tau}{2}\right)\tilde{\eta}^{k+1}(x) - \frac{\varepsilon\tau}{2h^2}[\tilde{\eta}^{k+1}(x+h) + \tilde{\eta}^{k+1}(x-h)]$$
$$\leqslant \left(1 - \frac{\varepsilon\tau}{h^2} + \frac{\tau}{2} - 2a\tau\xi(x)\right)\tilde{\eta}^k(x) + \frac{\varepsilon\tau}{2h^2}[\tilde{\eta}^k(x+h) + \tilde{\eta}^k(x-h)].$$

Clearly, $\tilde{\eta}^0(x) \leqslant 0$ for all $x \in \bar{I}_h$. Now suppose that for all $x \in \bar{I}_h$ and $j \leqslant k$, $\tilde{\eta}^j(x) \leqslant 0$. Assume

$$\tilde{\eta}^{k+1}(x^{(0)}) = \max_{x \in I_h} \tilde{\eta}^{k+1}(x).$$

Then

$$\left(1 - \frac{\tau}{2}\right)\tilde{\eta}^{k+1}(x^{(0)}) \leqslant \left(1 + \frac{\tau}{2} - 2aC_2\tau\right)\max_{x \in \bar{I}_h} \tilde{\eta}^k(x),$$

from which $\tilde{\eta}^{k+1}(x^{(0)}) \leqslant 0$ and so for all $x \in \bar{I}_h$, $\tilde{\eta}^{k+1}(x) \leqslant 0$. Thus the induction is completed.

Now we consider the following equation

$$\begin{cases} D(h, \tau, \varepsilon, a)u^k(x) = 0, & x \in I_h, \ k \geqslant 0, \\ u^k(x) = 0, & x = 0, 1, \ k \geqslant 0, \\ u^0(x) = U_0(x), & x \in \bar{I}_h. \end{cases} \qquad (A_2)$$

Lemma A₃. *Assume $0 \leqslant U_0(x) \leqslant C_3$, $C_4 = \max\left(C_3, \frac{1}{a}\right)$ and*

$$\tau \leqslant \tau_1 = \min\left(2, \frac{2h^2}{2\varepsilon + 4aC_4h^2 - h^2}\right).$$

Then for all $x \in \bar{I}_h$ and $k \geqslant 0$,

$$0 \leqslant u^k(x) \leqslant C_4.$$

Proof. Let $\eta^k(x) = u^k(x)$, $\xi^k(x) = C_4$. Applying Lemma A₂, we have $0 \leqslant u^k(x) \leqslant C_4$. On the other hand

$$\left(1 + \frac{\varepsilon\tau}{h^2} - \frac{\tau}{2}\right)u^{k+1}(x) - \frac{\varepsilon\tau}{2h^2}[u^{k+1}(x+h) + u^{k+1}(x-h)]$$
$$= \left(1 - \frac{\varepsilon\tau}{h^2} + \frac{\tau}{2} - a\tau u^k(x)\right)u^k(x) + \frac{\varepsilon\tau}{2h^2}[u^k(x+h) + u^k(x-h)]. \qquad (A_3)$$

Clearly $u^0(x) \geqslant 0$. Assume $\max_{x \in \bar{I}_h} u^j(x) \geqslant 0$ for $j \leqslant k$ and $u^{k+1}(x^{(0)}) = \min_{x \in \bar{I}_h} u^{k+1}(x)$. Then from (A₃) it follows that

$$\left(1-\frac{\tau}{2}\right)u^{k+1}(x^{(0)})\geqslant\left(1-\frac{\varepsilon\tau}{h^2}+\frac{\tau}{2}-a\tau C_4\right)u^k(x^{(0)})+\frac{\varepsilon\tau}{2h^2}[u^k(x^{(0)}+h)+u^k(x^{(0)}-h)]$$
$$\geqslant\left(1+\frac{\tau}{2}-a\tau C_4\right)\min_{x\in\bar{I}_h}u^k(x)\geqslant 0$$

and so for all $x\in\bar{I}_h$, $u^{k+1}(x)\geqslant 0$. The induction is completed.

Lemma A_4. *Let $u^k(x)$ be the solution of (A_2) and $U_0(x)$ be a supersolution of (A_1) with $f(z)=z(1-az)$, $0\leqslant U_0(x)\leqslant C_3$, $\tau\leqslant\tau_1$. Then $u^k(x)$ is a nonincreasing function of K for all $x\in\bar{I}_h$ and*

$$\lim_{k\to\infty}u^k(x)=v(x),$$

where $v(x)$ is the positive solution of (A_1). In particular, if $U_0(x)$ is a strict supersolution of (A_1), then $u^k(x)$ is strictly decreasing.

Proof. From Lemma A_3, we know that

$$0\leqslant u^k(x)\leqslant C_3,\quad x\in\bar{I}_h,\ k\geqslant 0.$$

Putting $\eta^k(x)=u^k(x)$ and $\xi^k(x)=U_0(x)$ in Lemma A_2, we get

$$0\leqslant u^k(x)\leqslant U_0(x).$$

In particular, $u^1(x)\leqslant U_0(x)$. Putting $\eta^k(x)=u^{k+1}(x)$ and $\xi^k(x)=u^k(x)$ in Lemma A_2, we obtain

$$0\leqslant u^{k+1}(x)\leqslant u^k(x),\quad x\in\bar{I}_h,\ k\geqslant 0.$$

Hence there is a function $\varphi(x)$ such that

$$\lim_{k\to\infty}u^k(x)=\varphi(x).$$

Let $k\to\infty$ in (A_2), and the first conclusion follows. Similarly, we get the second conclusion.

Similarly, we can prove the following result.

Lemma A_5. *Let $u^k(x)$ be the solution of (A_2) and $U_0(x)$ be a subsolution of (A_1) with $f(z)=z(1-az)$, $0\leqslant U_0(x)\leqslant C_3$, $\tau\leqslant\tau_1$. Then $u^k(x)$ is a nondecreasing function of k for all $x\in\bar{I}_h$ and*

$$\lim_{k\to\infty}u^k(x)=v(x),$$

where $v(x)$ is the positive solution of (A_1). In particular, if $U_0(x)$ is a strict subsolution of (A_1), then $u^k(x)$ is strictly increasing.

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