AN ECONOMICAL FINITE ELEMENT SCHEME FOR NAVIER-STOKES EQUATIONS*

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Abstract

In this paper, a new finite element scheme for Navier-Stokes equations is proposed, in which three different partitions (in the two dimensional case) are used to construct finite element subspaces of the velocity field and the pressure. The error estimate of the finite element approximation is given. The precision of this new scheme has the same order as the scheme Q_2/P_0 (biquadratic rectangular element for the velocity field, and constant rectangular element for the pressure), but it is more economical than the scheme Q_2/P_0.

§ 1. Introduction

In this paper we consider the boundary value problem of Navier-Stokes equations

$$-\nu\Delta\boldsymbol{u}+\sum_{j=1}^{2}u_j\frac{\partial\boldsymbol{u}}{\partial x_j}+\operatorname{grad}\lambda=\boldsymbol{f},\quad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div}\boldsymbol{u}=0,\quad \text{in } \Omega, \tag{1.2}$$

$$\boldsymbol{u}=0,\quad \text{on } \partial\Omega, \tag{1.3}$$

where $\Omega\subset\mathbf{R}^2$ is a bounded domain with a Lipschitz continuous boundary $\partial\Omega$, $\boldsymbol{u}=(u_1, u_2)$ is the velocity, λ is the pressure, ν is a positive constant which stands for the coefficient of kinematic viscosity, and $\boldsymbol{f}=(f_1, f_2)$ is given.

Let $W^{m,q}(\Omega)$ denote the Sobolev space on Ω with norm $\|\cdot\|_{m,q,\Omega}$. As usual, when $q=2$, $W^{m,2}(\Omega)$ is denoted by $H^m(\Omega)$ with norm $\|\cdot\|_{m,\Omega}$, and $W^{0,q}(\Omega)$ is denoted by $L^q(\Omega)$. Moreover, let $H_0^1(\Omega)=\{u\in H^1(\Omega),\ u=0 \text{ on } \partial\Omega\}$, $X=(H_0^1(\Omega))^2$ with norm $\|\cdot\|_X=|\cdot|_{1,\Omega}$ and $M=\left\{\lambda\in L^2(\Omega),\ \int_\Omega\lambda\,dx=0\right\}$ with norm $\|\cdot\|_M=\|\cdot\|_{0,\Omega}$. Then the boundary value problem (1.1)—(1.3) is equivalent to the following variational problem:

Find $(\boldsymbol{u},\lambda)\in X\times M$, such that

$$a_0(\boldsymbol{u},\boldsymbol{v})+a_1(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v})+b(\boldsymbol{v},\lambda)=(\boldsymbol{f},\boldsymbol{v}),\quad \forall\boldsymbol{v}\in X, \tag{1.4}$$

$$b(\boldsymbol{u},\mu)=0,\quad \forall\mu\in M, \tag{1.5}$$

where

$$a_0(\boldsymbol{u},\boldsymbol{v})=\nu\sum_{i,j=1}^{2}\int_\Omega\frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}dx, \tag{1.6}$$

$$a_1(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})=\frac{1}{2}\sum_{i,j=1}^{2}\int_\Omega w_j\left(\frac{\partial u_i}{\partial x_j}v_i-\frac{\partial v_i}{\partial x_j}u_i\right)dx, \tag{1.7}$$

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$$b(\boldsymbol{v},\lambda)=-\int_\Omega \lambda \operatorname{div} \boldsymbol{v}\, dx, \tag{1.8}$$

$$(\boldsymbol{f},\boldsymbol{v})=\sum_{i=1}^{2}\int_\Omega f_i v_i\, dx. \tag{1.9}$$

For the low Reynold's number, the variational problem (1.4)—(1.5) can be reduced to the Stokes problem:

Find $(\boldsymbol{u},\lambda)\in X\times M$, such that

$$a_0(\boldsymbol{u},\boldsymbol{v})+b(\boldsymbol{v},\lambda)=(\boldsymbol{f},\boldsymbol{v}),\quad \forall \boldsymbol{v}\in X, \tag{1.4*}$$

$$b(\boldsymbol{u},\mu)=0,\quad \forall \mu\in M. \tag{1.5}$$

Suppose X_h and M_h are two finite element subspaces of X and M. Consider the finite element approximation of (1.4)—(1.5) and (1.4)*—(1.5) respectively:

Find $(\boldsymbol{u}_h,\lambda_h)\in X_h\times M_h$, such that

$$a_0(\boldsymbol{u}_h,\boldsymbol{v}_h)+a_1(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{v}_h)+b(\boldsymbol{v}_h,\lambda_h)=(\boldsymbol{f},\boldsymbol{v}_h),\quad \forall \boldsymbol{v}_h\in X_h, \tag{1.10}$$

$$b(\boldsymbol{u}_h,\mu_h)=0,\quad \forall \mu_h\in M_h, \tag{1.11}$$

and

Find $(\boldsymbol{u}_h,\lambda_h)\in X_h\times M_h$, such that

$$a_0(\boldsymbol{u}_h,\boldsymbol{v}_h)+b(\boldsymbol{v}_h,\lambda_h)=(\boldsymbol{f},\boldsymbol{v}_h),\quad \forall \boldsymbol{v}_h\in X_h, \tag{1.10*}$$

$$b(\boldsymbol{u}_h,\mu_h)=0,\quad \forall \mu_h\in M_h. \tag{1.11}$$

The question we shall discuss is how to choose the finite element subspaces X_h and M_h, such that the error estimate of the finite element approximation $\{\boldsymbol{u}_h,\lambda_h\}$ is optimal. We know that if X_h and M_h are the optimal choice, then X_h and M_h should satisfy the following conditions[1-2]:

(a) The errors $\inf\limits_{\boldsymbol{v}_h\in X_h}\|\boldsymbol{u}-\boldsymbol{v}_h\|_X$ and $\inf\limits_{\mu_h\in M_h}\|\lambda-\mu_h\|_M$ have the same order in h, where $(\boldsymbol{u},\lambda)$ is the solution of (1.4)*—(1.5) or (1.4)—(1.5).

(b) There exists a constant β independent of h, such that

$$\sup_{\boldsymbol{v}_h\in X_h}\frac{b(\boldsymbol{v}_h,\mu_h)}{\|\boldsymbol{v}_h\|_X}\geqslant\beta\|\mu_h\|_M,\quad \forall \mu_h\in M_h. \tag{1.12}$$

Condition (1.12) is called Babuška–Brezzi condition.

In the case Ω is a rectangle, the domain Ω can be divided into some smaller rectangles. We shall denote by $\mathscr{T}_h$ this partition, and set P_k for the space of all polynomials of degree $\leqslant k$ in the variables x_1, x_2 and Q_k for the space of all polynomials of degree $\leqslant k$ with respect to each of the two variables x_1, x_2. J. T. Oden and O. Jacquotte[3] listed different choices of the subspaces X_h and M_h which satisfy the Babuška–Brezzi condition. For example, the Q_2/P_0 scheme (biquadratic rectangular element for the velocity field $\boldsymbol{u}$, and constant rectangular element for the pressure λ) is one of their choices. But in this scheme the error estimate of the finite element approximation $(\boldsymbol{u}_h,\lambda_h)$ is $\|\boldsymbol{u}-\boldsymbol{u}_h\|_X+\|\lambda-\lambda_h\|_M=O(h)$ only, even though they use the biquadratic rectangular element for velocity field $\boldsymbol{u}$. So it is interest to find a "one order precision scheme" with an optimal error estimate. O. A. Karakaskian[4] presented a scheme in which two different triangulations $\mathscr{T}_{\hat{h}}$ and $\mathscr{T}_h$ are used for approximating $\boldsymbol{u}$ and λ (linear triangular element for $\boldsymbol{u}$ and constant triangular element for λ to form subspaces X_h and M_h). He proved that if $\hat{h}/h$ is sufficiently small, subspaces X_h and M_h satisfy the Babuška–Brezzi condition and the optimal error estimate is given. But in practical application we do not know how to judge whether $\hat{h}/h$ is sufficiently small. For the rectangular element, J. T. Oden and O. Jacquotte[8] showed how to construct two partitions $\mathscr{T}_{\hat{h}}$ and $\mathscr{T}_h$ such that (a), (b) hold. Their idea is as follows. Suppose $\mathscr{T}_h$ is a given regular partition consisting of rectangles. For each rectangle $K \in \mathscr{T}_h$, connect the mid-points of the opposite sides of K; then each rectangle K is divided into four smaller rectangles. Let $\mathscr{T}_{\hat{h}}$ denote this new partition and use $\mathscr{T}_{\hat{h}}$ and $\mathscr{T}_h$ to form subspaces X_h and M_h respectively (bilinear rectangular element for u and constant rectangular element for λ). This scheme is denoted by $4Q_1/P_0$ and shown in Fig. 1.

Fig. 1

We know that the scheme $4Q_1/P_0$ satisfies the Babuška–Brezzi condition and the optimal error estimate (one order precision) is given. But if we compare scheme $4Q_1/P_0$ with scheme Q_2/P_0, we see that both schemes have one order precision and their resulting stiffness matrices have the same order and the same zero elements. Therefore we could not say which one is better even though the optimal error estimate is given for scheme $4Q_1/P_0$.

In this paper, we present a new scheme in which three different partitions are used for approximating u_1, u_2 and λ respectively. The optimal error estimate of this new scheme will be given. We will demonstrate that this new scheme is much more economical than scheme Q_2/P_0 and scheme $4Q_1/P_0$.

§ 2. Stokes Equations

In this section we only consider the Stokes problem (1.4)*—(1.5). Suppose $\mathscr{T}_h$ is a given regular partition of the rectangular domain Ω and $\Omega = \bigcup_{K \in \mathscr{T}_h} K$, where $h = \max_{K \in \mathscr{T}_h} \{h_K\}$ and h_K denotes the length of the longest side of K as shown in Fig. 2.

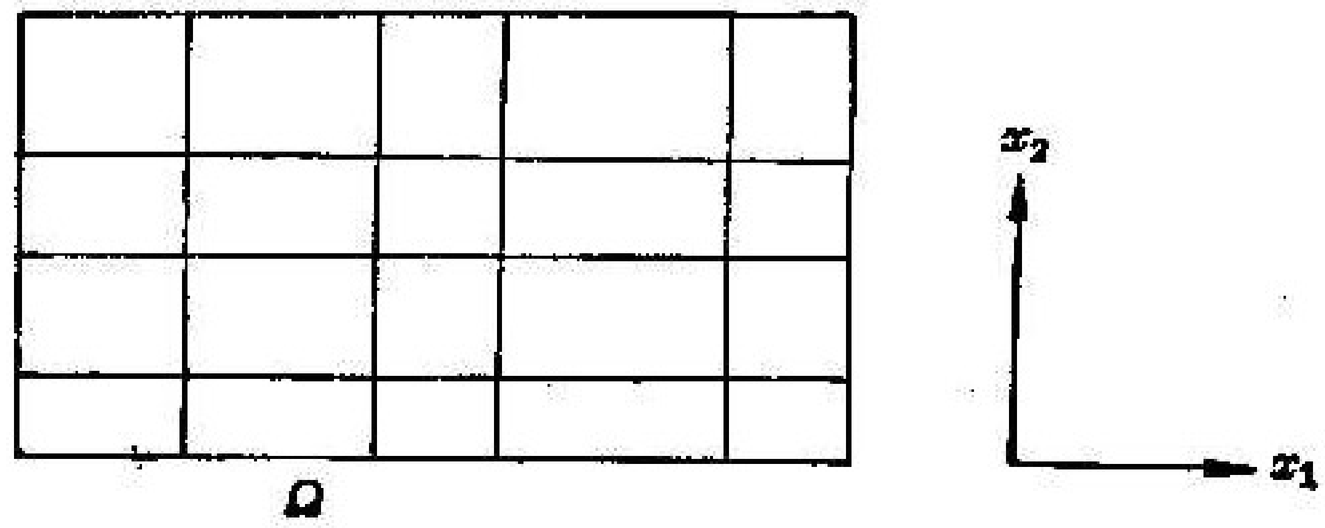

Fig. 2

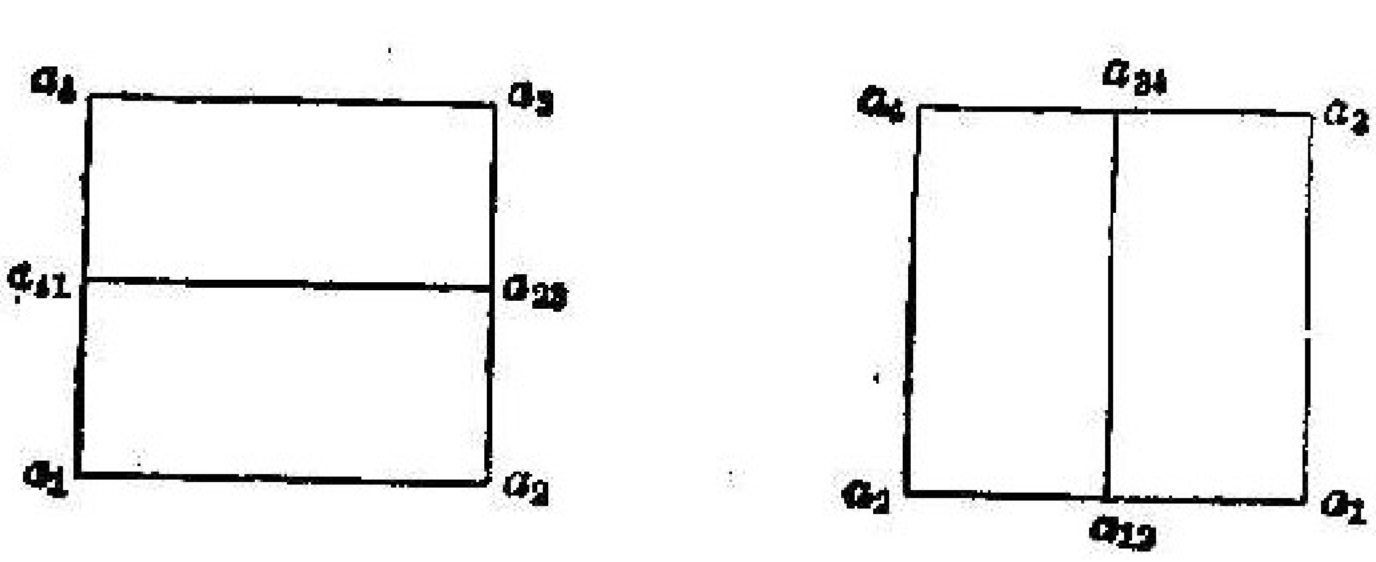

Fig. 3

Now we introduce two new partitions. For each rectangular element $K \in \mathscr{T}_h$, connect the mid-points of the opposite sides paralleling x_1-axis; then element K is divided into two smaller rectangles K_{11} and K_{12}. Therefore we obtain a new partition denoted by $\mathscr{T}_{h_1}$, and $\frac{h}{2} \leqslant h_1 \leqslant h$. Similarly, for each rectangular element $K \in \mathscr{T}_h$, connect the mid-points of the opposite sides paralleling x_2-axis; then K is divided into two smaller rectangles K_{21} and K_{22}. Hence we obtain another partition denoted by $\mathscr{T}_{h_2}$ with $\frac{h}{2} \leqslant h_2 \leqslant h$.

Using the partitions $\mathscr{T}_{h_1}$, $\mathscr{T}_{h_2}$ and $\mathscr{T}_h$, we construct subspaces X_h and M_h. Let

$$M_h=\left\{\mu_h \mid \mu_h|_K\in P_0(K),\ \forall K\in\mathscr{T}_h \text{ and } \int_\Omega \mu_h\,dx=0\right\},$$

$$S^1_{h_1}=\{v_h\in C(\overline{\Omega})\mid v_h|_{K_1}\in Q_1(K_1),\ \forall K_1\in\mathscr{T}_{h_1} \text{ and } v_h|_{\partial\Omega}=0\},$$

$$S^2_{h_2}=\{v_h\in C(\overline{\Omega})\mid v_h|_{K_2}\in Q_1(K_2),\ \forall K_2\in\mathscr{T}_{h_2} \text{ and } v_h|_{\partial\Omega}=0\},$$

$$X_h=S^1_{h_1}\times S^2_{h_2}.$$

Now we consider the discrete problem:

Find $(\boldsymbol{u}_h, \lambda_h)\in X_h\times M_h$, such that

$$a_0(\boldsymbol{u}_h, \boldsymbol{v}_h)+b(\boldsymbol{v}_h, \lambda_h)=(\boldsymbol{f}, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h\in X_h, \tag{2.1}$$

$$b(\boldsymbol{u}_h, \mu_h)=0, \quad \forall \mu_h\in M_h. \tag{2.2}$$

This scheme is denoted by $2Q_1/P_0$. In order to analyze the stability and convergence of scheme $2Q_1/P_0$, we introduce operators Π_1, Π_2 and Π as follows. For each $v_1\in H^1_0(\Omega)$, the following variational problem has a unique solution

$$(\nabla w^1_h, \nabla z_h)=(\nabla v_1, \nabla z_h), \quad \forall z_h\in S^1_{h_1}. \tag{2.3}$$

In fact, w^1_h is the projection of v_1 from $H^1_0(\Omega)$ to $S^1_{h_1}$. Define $\Pi_1 v_1\in S^1_{h_1}$ satisfying that on each element $K\in\mathscr{T}_h$,

$$\begin{cases}(\Pi_1 v_1)(a_i)=w^1_h(a_i), & 1\leqslant i\leqslant 4,\\ \displaystyle\int_{[a_i,a_j]}(\Pi_1 v_1-v_1)ds=0, & \text{where } [a_i, a_j]=[a_2, a_3] \text{ or } [a_4, a_1].\end{cases} \tag{2.4}$$

Obviously, $\Pi_1 v_1$ is uniquely determined by v_1. So we obtain operator Π_1: $H^1_0(\Omega)\to S^1_{h_1}$. Similarly, we define operator Π_2: $H^1_0(\Omega)\to S^2_{h_2}$ as follows. For every $v_2\in H^1_0(\Omega)$, $\Pi_2 v_2\in S^2_{h_2}$ satisfies that on each element $K\in\mathscr{T}_h$,

$$\begin{cases}(\Pi_2 v_2)(a_i)=w^2_h(a_i), & 1\leqslant i\leqslant 4,\\ \displaystyle\int_{[a_i,a_j]}(\Pi_2 v_2-v_2)ds=0, & \text{where } [a_i, a_j]=[a_1, a_2] \text{ or } [a_3, a_4],\end{cases} \tag{2.5}$$

where w^2_h is the projection of v_2 from $H^1_0(\Omega)$ to $S^2_{h_2}$.

For every $\boldsymbol{v}=(v_1, v_2)\in X$, we define

$$\Pi\boldsymbol{v}=(\Pi_1 v_1, \Pi_2 v_2)\in X_h.$$

It is clear that Π is an operator from X to X_h. Furthermore we have

Lemma 2.1. *The operator Π is bounded, namely there exists a positive constant C_1 independent of h_1, h_2 and h such that*

$$\|\Pi\boldsymbol{v}\|_X\leqslant C_1\|\boldsymbol{v}\|_X, \quad \forall \boldsymbol{v}\in X \tag{2.6}$$

and

$$b(\boldsymbol{v}-\Pi\boldsymbol{v}, \mu_h)=0, \quad \forall \boldsymbol{v}\in X, \forall \mu_h\in M_h. \tag{2.7}$$

Proof. For each $\boldsymbol{v}=(v_1, v_2)\in X$, let w^1_h, w^2_h denote the projections of v_1 and v_2 from $H^1_0(\Omega)$ to $S^1_{h_1}$ and $S^2_{h_2}$. Moreover, let $\Pi\boldsymbol{v}=\boldsymbol{v}_h=(v^1_h, v^2_h)$, $\boldsymbol{w}_h=(w^1_h, w^2_h)$, $\boldsymbol{e}_h=\boldsymbol{v}_h-\boldsymbol{w}_h$ and $\boldsymbol{e}=\boldsymbol{v}-\boldsymbol{w}_h=(e_1, e_2)$. Then we have

$$|\boldsymbol{v}_h|_{1,\Omega}\leqslant|\boldsymbol{w}_h|_{1,\Omega}+|\boldsymbol{e}_h|_{1,\Omega}.$$

Hence, we only need to estimate $|\boldsymbol{e}_h|_{1,\Omega}$ in order to get inequality (2.6). By $\boldsymbol{e}_h=(v^1_h-w^1_h, v^2_h-w^2_h)=(e^1_h, e^2_h)$ on each element $K\in\mathscr{T}_h$, we know that

$$\begin{cases} e_h^1(a_i)=0, \quad 1\leqslant i\leqslant 4, \\ \int_{[a_i,a_j]} [e_h^1-e_1]ds=0, \text{ where } [a_i, a_j]=[a_2, a_3] \text{ or } [a_4, a_1]. \end{cases} \tag{2.8}$$

From $(2.8)_2$, we have

$$\begin{cases} e_h^1(a_{23}) = \dfrac{1}{2|a_2a_3|}\displaystyle\int_{[a_2,a_3]} e_1\,dx_2, \\ e_h^1(a_{41}) = \dfrac{1}{2|a_1a_4|}\displaystyle\int_{[a_1,a_4]} e_1\,dx_2. \end{cases} \tag{2.9}$$

On the subelement K_{11} (see Fig. 3), we obtain

$$e_h^1(x)=e_h^1(a_{23})p_{23}(x)+e_h^1(a_{41})p_{41}(x), \quad \forall x\in K_{11}, \tag{2.10}$$

where $p_{23}(x)$, $p_{41}(x)\in Q_1(K_{11})$, $p_{23}(a_3)=p_{23}(a_4)=p_{23}(a_{41})=0$, $p_{23}(a_{23})=1$ and $p_{41}(a_3)=p_{41}(a_4)=p_{41}(a_{23})=0$, $p_{41}(a_{41})=1$. From (2.9), we have

$$\begin{cases} |e_h^1(a_{23})|\leqslant \dfrac{C}{h_K^{\frac{1}{2}}}\|e_1\|_{0,\partial K}\leqslant C(h_K^{-2}\|e_1\|_{0,K}^2+|e_1|_{1,K}^2)^{\frac{1}{2}}, \\ |e_h^1(a_{41})|\leqslant C(h_K^{-2}\|e_1\|_{0,K}^2+|e_1|_{1,K}^2)^{\frac{1}{2}}. \end{cases} \tag{2.11}$$

Combining (2.10) and (2.11), we obtain

$$|e_h^1|_{1,K_{11}}\leqslant |e_h^1(a_{23})|\,|p_{23}|_{1,K_{11}}+|e_h^1(a_{41})|\,|p_{41}|_{1,K_{11}}\leqslant C(h_K^{-2}\|e_1\|_{0,K}^2+|e_1|_{1,K}^2)^{\frac{1}{2}}.$$

Similarly, we have

$$|e_h^1|_{1,K_{12}}\leqslant C(h_K^{-2}\|e_1\|_{0,K}^2+|e_1|_{1,K}^2)^{\frac{1}{2}}.$$

Hence,

$$|e_h^1|_{1,K}^2\leqslant C(h_K^{-2}\|e_1\|_{0,K}^2+|e_1|_{1,K}^2),$$
$$|e_h^1|_{1,\Omega}^2\leqslant C(h^{-2}\|e_1\|_{0,\Omega}^2+|e_1|_{1,\Omega}^2). \tag{2.12}$$

By the Aubin-Nitsche technique, we know that

$$\|e_1\|_{0,\Omega}^2\leqslant Ch^2|e_1|_{1,\Omega}^2. \tag{2.13}$$

Combining (2.12) and (2.13), we get

$$|e_h^1|_{1,\Omega}\leqslant C|e_1|_{1,\Omega}\leqslant C|v_1|_{1,\Omega}.$$

Similarly, we can obtain the following inequality for e_h^2:

$$|e_h^2|_{1,\Omega}\leqslant C|v_2|_{1,\Omega}.$$

Therefore,

$$\|e_h\|_X\leqslant C\|v\|_X,$$

and inequality (2.6) follows immediately.

Finally, equality (2.7) follows directly from the definition of operator Π.

Lemma 2.2. *There exists a constant $\beta>0$, independent of h, such that*

$$\sup_{v_h\in X_h}\frac{b(v_h,\mu_h)}{\|v_h\|_X}\geqslant\beta\|\mu_h\|_M, \quad \forall\mu_h\in M_h.$$

Proof. For each $\mu_h\in M_h$, there exists a $v\in X$, such that[1]

$$\operatorname{div} v=-\mu_h \text{ and } \|v\|_X\leqslant C_0\|\mu_h\|_M, \ C_0>0.$$

Hence

$$\sup_{w_h\in X_h}\frac{b(w_h,\mu_h)}{\|w_h\|_X}\geqslant\frac{b(\Pi v,\mu_h)}{\|\Pi v\|_X}=\frac{b(v,\mu_h)}{\|\Pi v\|_X}=\frac{\|\mu_h\|_M^2}{\|\Pi v\|_X}\geqslant\frac{1}{C_0}\frac{\|v\|_X}{\|\Pi v\|_X}\|\mu_h\|_M.$$

The proof is completed with $\beta=\frac{1}{C_0C_1}$.

Lemma 2.3. *There exists a constant C_2 independent of h, u, λ, such that*

$$\inf_{v_h\in X_h}\|u-v_h\|_X\leqslant\|u-\Pi u\|_X\leqslant C_2h|u|_{2,\Omega},\quad\forall u\in X\cap(H^2(\Omega))^2,$$

$$\inf_{\mu_h\in M_h}\|\lambda-\mu_h\|_M\leqslant C_2h|\lambda|_{1,\Omega},\quad\forall\lambda\in M\cap H^1(\Omega).$$

The proof of this lemma can be found in [5].

An application of Theorem 1.1 in Chapter II of [1] yeilds the following error estimate.

Theorem 2.1. *The discrete problem* (2.1)—(2.2) *has a unique solution* $(u_h,\lambda_h)\in X_h\times M_h$ *and the following error estimate holds*

$$\|u-u_h\|_X+\|\lambda-\lambda_h\|_M\leqslant Ch\{|u|_{2,\Omega}+|\lambda|_{1,\Omega}\},\tag{2.14}$$

where (u,λ) *is the solution of problem* (1.4)*—(1.5) *and*

$$u\in X\cap(H^2(\Omega))^2,\quad\lambda\in M\cap H^1(\Omega).$$

The error estimate (2.14) shows that schemes $2Q_1/P_0$, $4Q_1/P_0$ and Q_2/P_0 have the same order of accuracy in h, but scheme $2Q_1/P_0$ is much more economical than schemes $4Q_1/P_0$ and Q_2/P_0.

Remark. In the case Ω is a general polygon, domain Ω can be divided into some triangles and some rectangles. On the triangular elements we use scheme P_2/P_0 and on the rectangular elements we still use scheme $2Q_1/P_0$ to construct subspaces X_h and M_h.

§ 3. Navier–Stokes Equations

Consider the finite element approximation of the nonlinear variational problem (1.4)—(1.5).

Find $(u_h,\lambda_h)\in X_h\times M_h$, such that

$$a_0(u_h,v_h)+a_1(u_h;u_h,v_h)+b(v_h,\lambda_h)=(f,v_h),\quad\forall v_h\in X_h,\tag{3.1}$$

$$b(u_h,\mu_h)=0,\quad\forall\mu_h\in M_h,\tag{3.2}$$

where subspaces X_h and M_h are given in the last section. Let

$$N=\sup_{u,v,w\in X}\frac{|a_1(w;u,v)|}{|u|_{1,\Omega}|v|_{1,\Omega}|w|_{1,\Omega}},$$

$$N_h=\sup_{u_h,v_h,w_h\in X_h}\frac{|a_1(w_h;u_h,v_h)|}{|u_h|_{1,\Omega}|v_h|_{1,\Omega}|w_h|_{1,\Omega}},$$

$$\|f\|^*=\sup_{v\in X}\frac{|(f,v)|}{|v|_{1,\Omega}},$$

$$\|f\|_h^*=\sup_{v_h\in X_h}\frac{|(f,v_h)|}{|v_h|_{1,\Omega}}.$$

Obviously, the following inequalities hold

$$N_h\leqslant N\text{ and }\|f\|_h^*\leqslant\|f\|^*.\tag{3.3}$$

For the problem (3.1)—(3.2), we have

Theorem 3.1. *Problem (3.1)—(3.2) has at least one solution* $(\boldsymbol{u}_h, \lambda_h)\in X_h\times M_h$, *which is unique if the following condition holds*

$$N_h\|\boldsymbol{f}\|_h^*/\nu^2<1.$$

The proof is omitted here; it is basically the same as in the continuous problem (see Theorems 1.2, 1.3, 1.4 in Chapter IV of [1]).

Theorem 3.2. *Suppose*

$$N\|\boldsymbol{f}\|^*/\nu^2\leqslant 1-\delta, \tag{3.4}$$

δ *is a constant*, $0<\delta<1$. *Then problem* (3.1)—(3.2) *has a unique solution* $(\boldsymbol{u}_h, \lambda_h)\in X_h\times M_h$ *and*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_X\leqslant C\left\{\|\boldsymbol{u}-\Pi\boldsymbol{u}\|_X+\inf_{\mu_h\in M_h}\|\lambda-\mu_h\|_M+\sup_{\boldsymbol{w}_h\in X_h}\frac{|G(\boldsymbol{u};\Pi\boldsymbol{u},\boldsymbol{w}_h)|}{\|\boldsymbol{w}_h\|_X}\right\}, \tag{3.5}$$

$$\|\lambda-\lambda_h\|_M\leqslant C\left\{\|\boldsymbol{u}-\Pi\boldsymbol{u}\|_X+\inf_{\mu_h\in M_h}\|\lambda-\mu_h\|_M+\sup_{\boldsymbol{w}_h\in X_h}\frac{|G(\boldsymbol{u};\Pi\boldsymbol{u},\boldsymbol{w}_h)|}{\|\boldsymbol{w}_h\|_X}\right.$$
$$\left.+\sup_{\boldsymbol{w}_h\in X_h}\frac{|G(\boldsymbol{u};\boldsymbol{u}_h,\boldsymbol{w}_h)|}{\|\boldsymbol{w}_h\|_X}\right\}, \tag{3.6}$$

where

$$G(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w})=a_1(\boldsymbol{u};\boldsymbol{u},\boldsymbol{w})-a_1(\boldsymbol{v};\boldsymbol{v},\boldsymbol{w}).$$

Proof. By inequalities (3.3), we know that

$$N_h\|\boldsymbol{f}\|_h^*/\nu^2\leqslant N\|\boldsymbol{f}\|^*/\nu^2<1-\delta.$$

From Theorem 3.1 we see that problem (3.1)—(3.2) has a unique solution $\{\boldsymbol{u}_h,\lambda_h\}\in X_h\times M_h$. Taking $\boldsymbol{v}_h=\boldsymbol{u}_h$ in (3.1), we obtain

$$a_0(\boldsymbol{u}_h,\boldsymbol{u}_h)=(\boldsymbol{f},\boldsymbol{u}_h).$$

Hence, we get

$$\|\boldsymbol{u}_h\|_X\leqslant\frac{|(\boldsymbol{f},\boldsymbol{u}_h)|}{\nu\|\boldsymbol{u}_h\|_X}\leqslant\frac{\|\boldsymbol{f}\|^*}{\nu}. \tag{3.7}$$

Let

$$\boldsymbol{z}_h=\boldsymbol{u}_h-\Pi\boldsymbol{u},$$
$$\varepsilon=a_0(\boldsymbol{u}_h,\boldsymbol{z}_h)+a_1(\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{z}_h)-a_0(\Pi\boldsymbol{u},\boldsymbol{z}_h)-a_1(\Pi\boldsymbol{u};\Pi\boldsymbol{u},\boldsymbol{z}_h).$$

Then we have

$$\begin{aligned}\varepsilon&=a_0(\boldsymbol{z}_h,\boldsymbol{z}_h)+a_1(\boldsymbol{z}_h;\boldsymbol{u}_h,\boldsymbol{z}_h)+a_1(\Pi\boldsymbol{u};\boldsymbol{z}_h,\boldsymbol{z}_h)\\&=a_0(\boldsymbol{z}_h,\boldsymbol{z}_h)+a_1(\boldsymbol{z}_h;\boldsymbol{u}_h,\boldsymbol{z}_h)\geqslant\nu\|\boldsymbol{z}_h\|_X^2-N\|\boldsymbol{u}_h\|_X\|\boldsymbol{z}_h\|_X^2\\&\geqslant\{\nu-N\|\boldsymbol{f}\|^*/\nu\}\|\boldsymbol{z}_h\|_X^2\geqslant\nu(1-\delta)\|\boldsymbol{z}_h\|_X^2.\end{aligned}$$

Namely, we obtain

$$\|\boldsymbol{z}_h\|_X^2\leqslant\frac{1}{\nu(1-\delta)}|\varepsilon|. \tag{3.8}$$

On the other hand, we know that

$$\varepsilon=b(\boldsymbol{z}_h,\lambda-\mu_h)+a_0(\boldsymbol{u}-\Pi\boldsymbol{u},\boldsymbol{z}_h)+G(\boldsymbol{u};\Pi\boldsymbol{u},\boldsymbol{z}_h),\quad\forall\mu_h\in M_h.$$

Furthermore, we obtain

$$|\varepsilon|\leqslant\left\{B\|\lambda-\mu_h\|_M+\nu\|\boldsymbol{u}-\Pi\boldsymbol{u}\|_X+\sup_{\boldsymbol{w}_h\in X_h}\frac{|G(\boldsymbol{u};\Pi\boldsymbol{u},\boldsymbol{w}_h)|}{\|\boldsymbol{w}_h\|_X}\right\}\|\boldsymbol{z}_h\|_X,\quad\forall\mu_h\in M_h. \tag{3.9}$$

Combining (3.8) and (3.9), we have

$$\|z_h\|_X \leqslant \frac{1}{\nu(1-\delta)}\left\{B \inf_{\mu_h\in M_h}\|\lambda-\mu_h\|_M+\nu\|u-\Pi u\|_X+\sup_{w_h\in X_h}\frac{|G(u;\,\Pi u,\,w_h)|}{\|w_h\|_X}\right\}.$$

The triangle inequality yields

$$\begin{aligned}\|u-u_h\|_X &\leqslant \|u-\Pi u\|_X+\|u_h-\Pi u\|_X\\ &\leqslant C\left\{\|u-\Pi u\|_X+\inf_{\mu_h\in M_h}\|\lambda-\mu_h\|_M+\sup_{w_h\in X_h}\frac{|G(u;\,\Pi u,\,w_h)|}{\|w_h\|_X}\right\},\end{aligned}$$

where C is a constant dependent only on B, δ, ν.

In order to estimate error $\|\lambda-\lambda_h\|_M$, we consider

$$\begin{aligned}b(v_h,\,\lambda_h-\mu_h) &= b(v_h,\,\lambda-\mu_h)-b(v_h,\,\lambda-\lambda_h)\\ &= b(v_h,\,\lambda-\mu_h)+a_0(u-u_h,\,v_h)+G(u;\,u_h,\,v_h),\end{aligned}$$

for every $v_h\in X_h$ and each $\mu_h\in M_h$. By Lemma 2.2, we obtain

$$\begin{aligned}\|\lambda_h-\mu_h\|_M &\leqslant \frac{1}{\beta}\sup_{v_h\in X_h}\frac{b(v_h,\,\lambda_h-\mu_h)}{\|v_h\|_X}\\ &\leqslant \frac{1}{\beta}\left\{B\|\lambda-\mu_h\|_M+\nu\|u-u_h\|_X+\sup_{v_h\in X_h}\frac{|G(u;\,u_h,\,v_h)|}{\|v_h\|_X}\right\},\quad \forall\mu_h\in M_h.\end{aligned} \tag{3.10}$$

Combining inequalities (3.10), (3.5) and the triangle inequality, the conclusion (3.6) follows immediately.

Now we need to estimate $G(u;\,\Pi u,\,v_h)$ and $G(u;\,u_h,\,v_h)$. We have

Lemma 3.1. *Suppose* $(u,\,\lambda)$ *is the solution of problem* (1.4)—(1.5) *and* $u\in X\cap(H^2(\Omega))^2$, $\lambda\in M\cap H^1(\Omega)$. *Then there exists a constant* C *independent of* h *and* $(u,\,\lambda)$, *such that*

$$\sup_{v_h\in X_h}\frac{|G(u;\,\Pi u,\,v_h)|}{\|v_h\|_X}\leqslant Ch\|u\|_X|u|_{2,\Omega}, \tag{3.11}$$

$$\sup_{v_h\in X_h}\frac{|G(u;\,u_h,\,v_h)|}{\|v_h\|_X}\leqslant C\|u-u_h\|_X\{\|f\|^*+\|u\|_X\}. \tag{3.12}$$

Proof. Since

$$\begin{aligned}|G(u;\,\Pi u,\,v_h)| &= |a_1(u;\,u,\,v_h)-a_1(\Pi u,\,\Pi u,\,v_h)|\\ &= |a_1(u-\Pi u;\,u,\,v_h)+a_1(\Pi u;\,u-\Pi u,\,v_h)|\\ &\leqslant N\|u-\Pi u\|_X\{\|u\|_X+\|\Pi u\|_X\}\|v_h\|_X\\ &\leqslant Ch\|u\|_X|u|_{2,\Omega}\|v_h\|_X,\end{aligned}$$

inequality (3.11) follows immediately.

Similarly, we have

$$\begin{aligned}|G(u;\,u_h,\,v_h)| &= |a_1(u-u_h;\,u,\,v_h)+a_1(u_h;\,u-u_h,\,v_h)|\\ &\leqslant N\|u-u_h\|_X\{\|u\|_X+\|u_h\|_X\}\|v_h\|_X\\ &\leqslant C\|u-u_h\|_X\{\|f\|^*+\|u\|_X\}\|v_h\|_X.\end{aligned}$$

Then inequality (3.12) is proved.

Finally, an application of Theorem 3.2 and Lemma 3.1 yields the following error estimates.

Theorem 3.3. *Suppose that condition* (3.4) *holds and that the solution of problem* (1.4)—(1.5) *satisfies* $u\in X\cap(H^2(\Omega))^2$, $\lambda\in M\cap H^1(\Omega)$. *Then the following error estimates hold*

$$\|u-u_h\|_X \leqslant Ch\{|u|_{2,\Omega}+|\lambda|_{1,\Omega}+|u|_{2,\Omega}\|u\|_X\}, \tag{3.13}$$

$$\|\lambda-\lambda_h\|_M \leqslant Ch\{|u|_{2,\Omega}+|\lambda|_{1,\Omega}+|u|_{2,\Omega}\|u\|_X\}(1+\|u\|_X+\|f\|^*). \tag{3.14}$$

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