SEVEN-DIAGONAL AND NINE-DIAGONAL SIP ALGORITHMS*

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§ 1. Introduction

It is well-known that the SIP method[1,2] is one of the best methods for solving systems of linear algebraic equations, but the convergence and the convergence rate of the method are a difficult problem. Moreover, till now only the case of five-diagonal matrix has been considered. In this paper we present the SIP algorithm for the case of seven-diagonal, nine-diagonal and more-diagonal matrices and consider its convergence and the rate of convergence. when the coefficient matrix is a diagonally dominant matrix.

§ 2. SIP Algorithm

Consider the matrix equation

$$Ax=f, \tag{1}$$

where

$$x=(x_{11}, \cdots, x_{1n}, \cdots, x_{m1}, \cdots, x_{mn})^T, \tag{2}$$

$$f=(f_{11}, \cdots, f_{1n}, \cdots, f_{m1}, \cdots, f_{mn})^T, \tag{3}$$

$$A=\begin{bmatrix} B_1 & C_1 & & \\ A_2 & B_2 & C_2 & \\ \cdots & \cdots & \cdots & \cdots \\ & & A_m & B_m \end{bmatrix}, \tag{4}$$

$$A_i=\begin{bmatrix} \eta_{i1} & \zeta_{i1} & & \\ \xi_{i2} & \eta_{i2} & \zeta_{i2} & \\ \cdots & \cdots & \cdots & \cdots \\ & & \xi_{in} & \eta_{in} \end{bmatrix}, \quad B_i=\begin{bmatrix} b_{i1} & c_{i1} & & \\ a_{i2} & b_{i2} & c_{i2} & \\ \cdots & \cdots & \cdots & \cdots \\ & & a_{in} & b_{in} \end{bmatrix},$$

$$C_i=\begin{bmatrix} \beta_{i1} & \gamma_{i1} & & \\ \alpha_{i2} & \beta_{i2} & \gamma_{i2} & \\ \cdots & \cdots & \cdots & \cdots \\ & & \alpha_{in} & \beta_{in} \end{bmatrix}. \tag{5}$$

Let

$$L=\begin{bmatrix} L_1 & & & \\ K_2 & \ddots & & \\ & \ddots & \ddots & \\ & & K_m & L_m \end{bmatrix}, \quad U=\begin{bmatrix} U_1 & V_1 & & \\ & \ddots & \ddots & \\ & & \ddots & V_{m-1} \\ & & & U_m \end{bmatrix}, \tag{6}$$

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where

$$L_i=\begin{bmatrix} l_{i1} & & \\ m_{i2} & \ddots & \\ & \ddots & \ddots \\ & & m_{in} & l_{in} \end{bmatrix},\quad K_i=\begin{bmatrix} e_{i1} & f_{i1} & & \\ d_{i2} & e_{i2} & f_{i2} & \\ \cdots\cdots & & & \\ \cdots\cdots & & & \\ & & d_{in} & e_{in} \end{bmatrix},$$

$$U_i=\begin{bmatrix} 1 & u_{i1} & & \\ & \ddots & \ddots & \\ & & \ddots & u_{in-1} \\ & & & 1 \end{bmatrix},\quad V_i=\begin{bmatrix} w_{i1} & t_{i1} & & \\ s_{i2} & w_{i2} & t_{i2} & \\ \cdots\cdots & & & \\ \cdots\cdots & & & \\ & & s_{in} & w_{in} \end{bmatrix}. \tag{7}$$

Therefore

$$LU=\begin{bmatrix} L_1U_1 & L_1V_1 & & \\ K_2U_1 & K_2V_1+L_2U_2 & L_2V_2 & \\ \cdots\cdots & & & \\ & & K_mU_{m-1} & K_mV_{m-1}+L_mU_m \end{bmatrix}, \tag{8}$$

where

$$K_iU_{i-1}=\begin{bmatrix} e_{i1} & e_{i1}u_{i-1,1}+f_{i1} & f_{i1}u_{i-1,2} & & \\ d_{i2} & d_{i2}u_{i-1,1}+e_{i2} & e_{i2}u_{i-1,2}+f_{i2} & f_{i2}u_{i-1,3} & \\ \cdots\cdots & & & & \\ & & & d_{in} & d_{in}u_{i-1,n-1}+e_{n} \end{bmatrix},$$

$$L_iV_i=\begin{bmatrix} l_{i1}w_{i1} & l_{i1}t_{i1} & & & \\ m_{i2}w_{i1}+l_{i2}s_{i2} & m_{i2}t_{i1}+l_{i2}w_{i2} & l_{i2}t_{i2} & & \\ m_{i3}s_{i2} & m_{i3}w_{i2}+l_{i3}s_{i3} & m_{i3}t_{i2}+l_{i3}w_{i3} & l_{i3}t_{i3} & \\ \cdots\cdots & & & & \\ & & m_{in}s_{in-1} & m_{in}w_{in-1}+l_{in}s_{in} & m_{in}t_{in-1}+l_{in}w_{in} \end{bmatrix},$$

$$K_iV_{i-1}=\begin{bmatrix} e_{i1}w_{i-1,1}+f_{i1}s_{i-1,2} & e_{i1}t_{i-1,1}+f_{i1}w_{i-1,2} & f_{i1}t_{i-1,2} & & & \\ d_{i2}w_{i-1,1}+e_{i2}s_{i-1,2} & d_{i2}t_{i-1,1}+e_{i2}w_{i-1,2}+f_{i2}s_{i-1,3} & e_{i2}t_{i-1,2}+f_{i2}w_{i-1,3} & f_{i2}t_{i-1,3} & & \\ d_{i3}s_{i-1,2} & d_{i3}w_{i-1,2}+e_{i3}s_{i-1,3} & d_{i3}t_{i-1,2}+e_{i3}w_{i-1,3}+f_{i3}s_{i-1,4} & e_{i3}t_{i-1,3}+f_{i3}w_{i-1,4} & f_{i3}t_{i-1,4} & \\ \cdots\cdots & & & & & \\ & & & d_{in}s_{i-1,n-1} & d_{in}t_{i-1,n-1}+e_{in}w_{i-1,n} & \\ & & & & & d_{in}w_{i-1,n-1}+e_{in}s_{i-1,n} \end{bmatrix}, \tag{9}$$

$$L_iU_i=\begin{bmatrix} l_{i1} & l_{i1}u_{i1} & & \\ m_{i2} & m_{i2}u_{i1}+l_{i2} & l_{i2}u_{i2} & \\ & m_{i3} & m_{i3}u_{i2}+l_{i3} & l_{i3}u_{i3} \\ \cdots\cdots & & & \\ & & m_{in} & m_{in}u_{i,n-1}+l_{in} \end{bmatrix}.$$

If $M=LU$ is the complete decomposition of A, i.e. $LU=A$, we have

$$\begin{cases} d_{ij}=\xi_{ij}, \\ d_{ij}u_{i-1,j-1}+e_{ij}=\eta_{ij}, \\ e_{ij}u_{i-1,j}+f_{ij}=\zeta_{ij}, \\ f_{ij}u_{i-1,j+1}=0, \end{cases}$$

$$\begin{cases} d_{ij}s_{i-1,j-1}=0, \\ d_{ij}w_{i-1,j-1}+e_{ij}s_{i-1,j}+m_{ij}=a_{ij}, \\ d_{ij}t_{i-1,j-1}+e_{ij}w_{i-1,j}+f_{ij}s_{i-1,j+1}+m_{ij}u_{i,j-1}+l_{ij}=b_{ij}, \\ e_{ij}t_{i-1,j}+f_{ij}w_{i-1,j+1}+l_{ij}u_{ij}=c_{ij}, \\ f_{ij}t_{i-1,j+1}=0, \end{cases} \tag{10}$$

$$\begin{cases} m_{ij}s_{i,j-1}=0, \\ m_{ij}w_{i,j-1}+l_{ij}s_{ij}=\alpha_{ij}, \\ m_{ij}t_{i,j-1}+l_{ij}w_{ij}=\beta_{ij}, \\ l_{ij}t_{ij}=\gamma_{ij}. \end{cases}$$

Here, we assume that the quantities which do not appear in (4), (5), (6) and (7) are zeros. Obviously, the number of equations is more than that of the unknowns (the unknown elements of L and U). In general, we cancel from (10) those equations whose RHS (right-hand side) are zeros. Then, we add some linear combinations of the LHS (left-hand side) of these equations to the RHS of the other equations. Thus, we obtain

$$\begin{cases} d_{ij}=\xi_{ij}+\phi^d_{ij}, \\ d_{ij}u_{i-1,j-1}+e_{ij}=\eta_{ij}+\phi^e_{ij}, \\ e_{ij}u_{i-1,j}+f_{ij}=\zeta_{ij}+\phi^f_{ij}, \end{cases}$$

$$\begin{cases} d_{ij}w_{i-1,j-1}+e_{ij}s_{i-1,j}+m_{ij}=a_{ij}+\phi^m_{ij}, \\ d_{ij}t_{i-1,j-1}+e_{ij}w_{i-1,j}+f_{ij}s_{i-1,j+1}+m_{ij}u_{i,j-1}+l_{ij}=b_{ij}+\phi^l_{ij}, \\ e_{ij}t_{i-1,j}+f_{ij}w_{i-1,j+1}+l_{ij}u_{ij}=c_{ij}+\phi^u_{ij}, \end{cases} \tag{11}$$

$$\begin{cases} m_{ij}w_{i,j-1}+l_{ij}s_{ij}=\alpha_{ij}+\phi^s_{ij}, \\ m_{ij}t_{i,j-1}+l_{ij}w_{ij}=\beta_{ij}+\phi^w_{ij}, \\ l_{ij}t_{ij}=\gamma_{ij}+\phi^t_{ij}, \end{cases}$$

where ϕ's are the linear combinations mentioned above. The number of the equations is just the same as that of the unknowns. Thus, we can solve the equations and obtain the matrices L and U. Finally, we use the iterative formula

$$Mx^{(s+1)}=Nx^{(s)}+f, \quad s=0, 1, \cdots, \tag{12}$$

to solve equations (1), where $M=LU$, $N=M-A$. We choose the linear combinations ϕ's such that the convergence of (12) is good and the system of equations (11) can easily be solved. One of our choices is

$$\begin{gathered} \phi^d_{ij}=\theta^d d_{ij}s_{i-1,j-1}, \quad \phi^e_{ij}=-\theta^d d_{ij}s_{i-1,j-1}, \quad \phi^f_{ij}=-f_{ij}u_{i-1,j+1}, \\ \phi^m_{ij}=-d_{ij}s_{i-1,j-1}, \quad \phi^l_{ij}=0, \quad \phi^u_{ij}=-f_{ij}t_{i-1,j+1}, \\ \phi^s_{ij}=-m_{ij}s_{i,j-1}, \quad \phi^w_{ij}=-\theta^w f_{ij}t_{i-1,j+1}, \quad \phi^t_{ij}=\theta^w f_{ij}t_{i-1,j+1}, \end{gathered} \tag{13}$$

where θ^d and θ^w are certain parameters. With these quantities we can solve (11) recursively, i.e. for each $i=1, 2, \cdots, m$, we compute the unknowns d_{ij}, e_{ij}, f_{ij}, m_{ij}, l_{ij}, u_{ij}, s_{ij}, w_{ij} and t_{ij} from 1 to n recursively. Here we assume the quantities which do not appear in (4), (5), (6) and (7) are zeros. The algorithm (11), (12) is a nine-diagonal SIP algorithm. Now we consider the seven-diagonal SIP algorithm. There are two cases:

1.
$$\xi_{ij}\equiv\gamma_{ij}\equiv 0. \tag{14}$$

In this case, we take

$$d_{ij}\equiv t_{ij}\equiv 0 \tag{15}$$

and (11) becomes

$$\begin{gathered} e_{ij}=\eta_{ij},\quad e_{ij}u_{i-1,j}+f_{ij}=\zeta_{ij},\quad f_{ij}u_{i-1,j+1}=0,\\ e_{ij}s_{i-1,j}+m_{ij}=a_{ij},\quad e_{ij}w_{i-1,j}+f_{ij}s_{i-1,j+1}+m_{ij}u_{i,j-1}+l_{ij}=b_{ij},\\ f_{ij}w_{i-1,j+1}+l_{ij}u_{ij}=c_{ij},\\ m_{ij}s_{i,j-1}=0,\quad m_{ij}w_{i,j-1}+l_{ij}s_{ij}=\alpha_{ij},\quad l_{ij}w_{ij}=\beta_{ij}. \end{gathered} \tag{16}$$

As before, we cancel those equations whose RHS are zeros. i.e.

$$f_{ij}u_{i-1,j+1}=0,\quad m_{ij}s_{i,j-1}=0, \tag{17}$$

and add some linear combinations of $f_{ij}u_{i-1,j+1}$ and $m_{ij}s_{i,j-1}$ to the RHS of the other equations. We obtain

$$\begin{gathered} e_{ij}=\eta_{ij}+\phi^e_{ij},\quad e_{ij}u_{i-1,j}+f_{ij}=\zeta_{ij}+\phi^f_{ij},\\ e_{ij}s_{i-1,j}+m_{ij}=a_{ij}+\phi^m_{ij},\quad e_{ij}w_{i-1,j}+f_{ij}s_{i-1,j+1}+m_{ij}u_{i,j-1}+l_{ij}=b_{ij}+\phi^l_{ij},\\ f_{ij}w_{i-1,j+1}+l_{ij}u_{ij}=c_{ij}+\phi^u_{ij},\quad m_{ij}w_{i,j-1}+l_{ij}s_{ij}=\alpha_{ij}+\phi^s_{ij},\\ l_{ij}w_{ij}=\beta_{ij}+\phi^w_{ij}. \end{gathered} \tag{18}$$

We call this algorithm the 7_{II} SIP algorithm.

One of our choices for the ϕ's is

$$\begin{gathered} \phi^e_{ij}=0,\quad \phi^f_{ij}=-f_{ij}u_{i-1,j+1},\quad \phi^m_{ij}=-\theta^m m_{ij}s_{i,j-1},\\ \phi^l_{ij}=\theta^m m_{ij}s_{i,j-1}+\theta^u f_{ij}u_{i-1,j+1},\quad \phi^u_{ij}=-\theta^u f_{ij}u_{i-1,j+1},\\ \phi^s_{ij}=-m_{ij}s_{i,j-1},\quad \phi^w_{ij}=0, \end{gathered} \tag{19}$$

with these quantities we can solve (18) recursively.

2.
$$\zeta_{ij}\equiv\alpha_{ij}\equiv 0. \tag{20}$$

In this case, we take

$$f_{ij}\equiv s_{ij}\equiv 0 \tag{21}$$

and (11) becomes

$$\begin{gathered} d_{ij}=\xi_{ij},\quad d_{ij}u_{i-1,j-1}+e_{ij}=\eta_{ij},\quad e_{ij}u_{i-1,j}=0,\\ d_{ij}w_{i-1,j-1}+m_{ij}=a_{ij},\quad d_{ij}t_{i-1,j-1}+e_{ij}w_{i-1,j}+m_{ij}u_{i,j-1}+l_{ij}=b_{ij},\\ e_{ij}t_{i-1,j}+l_{ij}u_{ij}=c_{ij},\\ m_{ij}w_{i,j-1}=0,\ m_{ij}t_{i,j-1}+l_{ij}w_{ij}=\beta_{ij},\ l_{ij}t_{ij}=\gamma_{ij}. \end{gathered} \tag{22}$$

We proceed as before and obtain

$$\begin{gathered} d_{ij}=\xi_{ij}+\phi^d_{ij},\ d_{ij}u_{i-1,j-1}+e_{ij}=\eta_{ij}+\phi^e_{ij},\\ d_{ij}w_{i-1,j-1}+m_{ij}=a_{ij}+\phi^m_{ij}, \end{gathered}$$

$$d_{ij}t_{i-1,j-1}+e_{ij}w_{i-1,j}+m_{ij}u_{i,j-1}+l_{ij}=b_{ij}+\phi^l_{ij},$$
$$e_{ij}t_{i-1,j}+l_{ij}u_{ij}=c_{ij}+\phi^u_{ij}, \tag{23}$$
$$m_{ij}t_{i,j-1}+l_{ij}w_{ij}=\beta_{ij}+\phi^w_{ij}, \quad l_{ij}t_{ij}=\gamma_{ij}+\phi^t_{ij},$$

where ϕ^d_{ij}, ϕ^e_{ij} etc. are some linear combinations of $e_{ij}u_{i-1,j}$ and $m_{ij}w_{i,j-1}$, we call this algorithm the 7_I SIP algorithm. A choice of ϕ's is

$$\phi^d_{ij}=0, \quad \phi^e_{ij}=-e_{ij}u_{i-1,j},$$
$$\phi^m_{ij}=-\theta^m m_{ij}w_{i,j-1}, \; \phi^l_{ij}=\theta^m m_{ij}w_{i,j-1}+\theta^u e_{ij}u_{i-1,j},$$
$$\phi^u_{ij}=-\theta^u e_{ij}u_{i-1,j},$$
$$\phi^w_{ij}=-m_{ij}w_{i,j-1}, \quad \phi^t_{ij}=0. \tag{24}$$

With these quantities we can solve (23) recursively.

In particular, if A is a five-diagonal matrix, i.e.

$$\xi_{ij}\equiv\zeta_{ij}\equiv\alpha_{ij}\equiv\gamma_{ij}\equiv 0, \tag{25}$$

we take

$$d_{ij}\equiv f_{ij}\equiv s_{ij}\equiv t_{ij}\equiv 0 \tag{26}$$

and (11) becomes

$$e_{ij}=\eta_{ij}, \quad e_{ij}u_{i-1,j}=0,$$
$$m_{ij}=a_{ij}, \; e_{ij}w_{i-1,j}+m_{ij}u_{i,j-1}+l_{ij}=b_{ij}, \; l_{ij}u_{ij}=c_{ij},$$
$$m_{ij}w_{i,j-1}=0, \quad l_{ij}w_{ij}=\beta_{ij}.$$

To determine L and U, we solve the system of equations

$$e_{ij}=\eta_{ij}+\phi^e_{ij}, \quad m_{ij}=a_{ij}+\phi^m_{ij},$$
$$e_{ij}w_{i-1,j}+m_{ij}u_{i,j-1}+l_{ij}=b_{ij}+\phi^l_{ij}, \tag{27}$$
$$l_{ij}u_{ij}=c_{ij}+\phi^u_{ij}, \; l_{ij}w_{ij}=\beta_{ij}+\phi^w_{ij},$$

where ϕ^e_{ij}, ϕ^m_{ij}, ϕ^l_{ij}, ϕ^u_{ij} and ϕ^w_{ij} are some linear combinations of $e_{ij}u_{i-1,j}$ and $m_{ij}w_{i,j-1}$. For example,

$$\phi^e_{ij}=-\theta^e e_{ij}u_{i-1,j}, \quad \phi^m_{ij}=-\theta^m m_{ij}w_{i,j-1},$$
$$\phi^l_{ij}=\theta^m m_{ij}w_{i,j-1}+\theta^u e_{ij}u_{i-1,j},$$
$$\phi^u_{ij}=-\theta^u e_{ij}u_{i-1,j}, \quad \phi^w_{ij}=-\theta^w m_{ij}w_{i,j-1}. \tag{28}$$

With these ϕ's, (27) is the classical five-diagonal SIP algorithm.

In general, if A is a matrix with an arbitrary number of diagonals different from zero, we can proceed similarly and obtain SIP algorithms of other types.

§ 3. The convergence of SIP algorithm

In general, it is very difficult to determine whether a certain SIP algorithm is convergent or not. Here, we assume that the matrix A is a diagonally dominant matrix as it is the case in many problems. Without loss of generality, we may assume the diagonal elements of A to be one, i.e. in (5)

$$b_{ij}\equiv 1. \tag{29}$$

Otherwise, we may consider the equivalent equation

$$D^{-1}Ax=D^{-1}f,$$

where $D=\text{diag}\, A$. Under the condition that A is diagonally dominant, the quantities a_{ij}, c_{ij}, ξ_{ij}, η_{ij}, ζ_{ij}, α_{ij}, β_{ij} and γ_{ij} are all less than one in absolute value.

Thus we may assume

$$\begin{gathered}\xi_{ij}=O(k), \quad \eta_{ij}=O(k), \quad \zeta_{ij}=O(k),\\ a_{ij}=O(h), \quad c_{ij}=O(h),\\ \alpha_{ij}=O(k), \quad \beta_{ij}=O(k), \quad \gamma_{ij}=O(k),\end{gathered} \tag{30}$$

where h and k are small. Now, in order to consider the convergence of the iterative formula (12), it is necessary to consider the spectral radius $\rho(M^{-1}N)$ of the matrix $M^{-1}N$, but

$$M^{-1}N=(A+N)^{-1}N=(I+A^{-1}N)^{-1}A^{-1}N,$$

and if $\|A^{-1}N\|<1$, where $\|G\|$ is an arbitrary norm of the matrix G, we have

$$\|(I+A^{-1}N)^{-1}\|\leqslant 1+\|A^{-1}N\|+\|A^{-1}N\|^2+\cdots=\frac{1}{(1-\|A^{-1}N\|)}.$$

Hence

$$\rho(M^{-1}N)\leqslant\|M^{-1}N\|\leqslant\|A^{-1}N\|/(1-\|A^{-1}N\|). \tag{31}$$

If we denote the elements of A by a_{ij} and those of N by n_{ij}, under the condition that A is strictly diagonally dominant, from [3] we have

$$\|A^{-1}N\|_\infty\leqslant\max_i\Big[\sum_j|n_{ij}|/\Big(|a_{ii}|-\sum_{j\neq i}|a_{ij}|\Big)\Big]. \tag{32}$$

From (31) and (32), we can immediately obtain an upper bound of $\rho(M^{-1}N)$.

Now, we consider the nine-diagonal SIP algorithm (11), (12). If h and k are small, it can easily be verified by mathematical induction that

$$\begin{gathered}d_{ij}=\xi_{ij}+O(h^2+k^2), \quad e_{ij}=\eta_{ij}+O(h^2+k^2), \quad f_{ij}=\zeta_{ij}+O(h^2+k^2),\\ m_{ij}=a_{ij}+O(h^2+k^2), \quad l_{ij}=1+O(h^2+k^2), \quad u_{ij}=c_{ij}+O(h^2+k^2),\\ s_{ij}=\alpha_{ij}+O(h^2+k^2), \quad w_{ij}=\beta_{ij}+O(h^2+k^2), \quad t_{ij}=\gamma_{ij}+O(h^2+k^2).\end{gathered} \tag{33}$$

But if

$$N=\begin{bmatrix}N_1^B & N_1^C & & \\ N_2^A & N_2^B & N_2^C & \\ \cdots & \cdots & \cdots & \cdots\\ & & N_m^A & N_m^B\end{bmatrix}, \tag{34}$$

where

$$N_i^A=\begin{bmatrix}n_{i1}^\eta & n_{i1}^\zeta & n_{i1}^{\zeta+} & \\ n_{i2}^\xi & n_{i2}^\eta & n_{i2}^\zeta & n_{i2}^{\zeta+}\\ \cdots & \cdots & \cdots & \cdots\\ & & n_{in}^\xi & n_{in}^\eta\end{bmatrix}, \quad N_i^C=\begin{bmatrix}n_{i1}^\beta & n_{i1}^\gamma & & \\ n_{i2}^\alpha & n_{i2}^\beta & n_{i2}^\gamma & \\ \cdots & \cdots & \cdots & \cdots\\ & n_{in}^{\alpha-} & n_{in}^\alpha & n_{in}^\beta\end{bmatrix},$$

$$N_i^B=\begin{bmatrix}n_{i1}^b & n_{i1}^c & n_{i1}^{c+} & \\ n_{i2}^a & n_{i2}^b & n_{i2}^c & n_{i2}^{c+}\\ \cdots & \cdots & \cdots & \cdots\\ & n_{in}^{a-} & n_{in}^a & n_{in}^b\end{bmatrix}, \tag{35}$$

from (8), (9), (4) and (5) we have

$$
\begin{gathered}
n_{ij}^{\xi}=\phi_{ij}^{d},\ n_{ij}^{\eta}=\phi_{ij}^{e},\ n_{ij}^{\zeta}=\phi_{ij}^{f},\ n_{ij}^{\zeta+}=f_{ij}u_{i-1,j+1},\\
n_{ij}^{a+}=d_{ij}s_{i-1,j-1},\quad n_{ij}^{a}=\phi_{ij}^{m},\quad n_{ij}^{b}=\phi_{ij}^{l},\\
n_{ij}^{c}=\phi_{ij}^{u},\quad n_{ij}^{c+}=f_{ij}t_{i-1,j+1},\\
n_{ij}^{\alpha-}=m_{ij}s_{i,j-1},\quad n_{ij}^{\alpha}=\phi_{ij}^{s},\quad n_{ij}^{\beta}=\phi_{ij}^{w},\quad n_{ij}^{\gamma}=\phi_{ij}^{t}.
\end{gathered}
\tag{36}
$$

Since the ϕ's are some linear combinations of $f_{ij}u_{i-1,j+1}$, $d_{ij}s_{i-1,j-1}$, $f_{ij}t_{i-1,j+1}$ and $m_{ij}s_{i,j-1}$, it follows from (33), (34), (35) and (36) that

$$n_{ij}^{\xi}=O(h^2+k^2),\ n_{ij}^{\eta}=O(h^2+k^2),\ \cdots,\ n_{ij}^{\gamma}=O(h^2+k^2).$$

We denote these by the notation

$$N=O(h^2+k^2), \tag{37}$$

which represents that the elements of the matrix N are $O(h^2+k^2)$. On the other hand, due to (29) and (30),

$$|a_{ii}|-\sum_{j\neq i}|a_{ij}|=1-O(h+k).$$

It follows from (32), that

$$\|A^{-1}N\|_\infty\leqslant O(h^2+k^2).$$

Hence, from (31) we have

$$\rho(M^{-1}N)\leqslant O(h^2+k^2). \tag{38}$$

In particular, if the ϕ's are taken as in (13), it may be seen that

$$
\begin{gathered}
n_{ij}^{\xi}=O(k^2),\quad n_{ij}^{\eta}=O(k^2),\quad n_{ij}^{\zeta}=O(hk),\quad n_{ij}^{\zeta+}=O(hk),\\
n_{ij}^{a-}=O(k^2),\quad n_{ij}^{a}=O(k^2),\quad n_{ij}^{b}=O,\quad n_{ij}^{c}=O(k^2),\quad n_{ij}^{c+}=O(k^2),\\
n_{ij}^{\alpha-}=O(hk),\quad n_{ij}^{\alpha}=O(hk),\quad n_{ij}^{\beta}=O(k^2),\quad n_{ij}^{\gamma}=O(k^2).
\end{gathered}
\tag{39}
$$

Hence

$$\rho(M^{-1}N)\leqslant O(k^2+hk). \tag{40}$$

The estimates (38) and (40) imply that if A is diagonally dominant and the off-diagonal elements of A are sufficiently small with respect to the diagonal elements on the same row, the nine-diagonal SIP algorithm (11), (12) is convergent and the estimate (38) holds.

Now, we consider the case (14) and the seven-diagonal SIP algorithm (18), (12). Under the conditions (29) and (30), we have

$$f_{ij}u_{i-1,j+1}=O(hk),\quad m_{ij}s_{i,j-1}=O(hk).$$

We proceed as before. In place of (38), we obtain the estimate

$$\rho(M^{-1}N)\leqslant O(hk), \tag{41}$$

since those ϕ's are some linear combinations of $f_{ij}u_{i-1,j+1}$ and $m_{ij}s_{i,j-1}$.

Similarly, if A is a seven-diagonal matrix with (20) and we apply the seven-diagonal SIP algorithm (23), (12), we have also the estimate (41).

Finally, for the five-diagonal matrix with (25) and the five-diagonal SIP algorithm (27), (12), the estimate (41) also holds.

Now, we may use the estimates obtained above to compare the SIP method with other methods for solving systems of linear algebraic equations. For example, if we apply the estimate

$$\|M^{-1}N\|_\infty\leqslant\max_i\Big[\sum_j|n_{ij}|/\Big(|m_{ii}|-\sum_{j\neq i}|m_{ij}|\Big)\Big] \tag{42}$$

in [3] to the Jacobi method (J method) or the Gauss-Seidel method (GS method), we can only obtain the estimate

$$\rho(M^{-1}N) \leqslant O(h+k),$$

which is worse than that for the SIP method. Computational results also show that the SIP method is much better than the J method and the GS method and it is also better than the SOR method. Moreover, when A is a seven-diagonal or nine-diagonal matrix, it is very difficult to find the optimum relaxation factor.

The SIP method with diagonals other than five, seven and nine can be considered analogously and similar results can be obtained.

§ 4. SIP Method of Higher Accuracy

In § 3, we take the diagonals of L and U, on which the elements are different from zero, to be the same as those of A. In practice, we may also take the former to be different from the latter. In particular, the former may involve the latter in it. For example, we may apply the seven-diagonal or nine-diagonal SIP algorithm to a five-diagonal matrix A and the nine-diagonal SIP algorithm to a seven-diagonal matrix A. The rate of convergence of such a SIP algorithm may be better than that of the corresponding SIP algorithm in § 2. Let us consider it here.

For convenience, in this section, we assume that (29) holds and all the quantities on the LHS of (30) are $O(h)$. Now, if A is a five-diagonal matrix with (25) and we apply the seven-diagonal SIP algorithm (18), (12), under our conditions, the estimates (33) with $O(h^2+k^2)$ replaced by $O(h^2)$ hold. But due to (14), we have

$$f_{ij}=O(h^2), \quad s_{ij}=O(h^2).$$

Thus,

$$f_{ij}u_{i-1,j+1}=O(h^3), \quad m_{ij}s_{i,j-1}=O(h^3). \tag{43}$$

Hence, instead of the estimate (41) which becomes

$$\rho(M^{-1}N) \leqslant O(h^2) \tag{44}$$

now we have

$$\rho(M^{-1}N) \leqslant O(h^3). \tag{45}$$

From this, we conclude that the rate of convergence is modified when we apply the seven-diagonal SIP algorithm (18), (12) instead of the five-diagonal SIP algorithm to the five-diagonal matrix A.

Now, if A is a five-diagonal matrix mentioned above and we apply the seven-diagonal SIP algorithm (20), (12), we obtain

$$e_{ij}u_{i-1,j}=O(h^2), \quad m_{ij}w_{i,j-1}=O(h^2).$$

Thus, we still have the estimate (44). In other words, the rate of convergence can not be modified.

If we apply the nine-diagonal SIP algorithm (11), (12) to the above five-diagonal matrix A, since the LHS of (36) are all the linear combinations of $f_{ij}u_{i-1,j+1}$, $d_{ij}s_{i-1,j-1}$, $f_{ij}t_{i-1,j+1}$ and $m_{ij}s_{i,j-1}$ and

$$f_{ij}u_{i-1,j+1}=O(h^3), \quad d_{ij}s_{i-1,j-1}=O(h^4),$$
$$f_{ij}t_{i-1,j+1}=O(h^4), \quad m_{ij}s_{i,j-1}=O(h^3),$$

we also have the estimate (45). This shows that to solve the system of equations (1) where A is the familiar five-diagonal matrix, the use of the nine-diagonal SIP algorithm (11), (12) is perhaps no better than the use of the seven-diagonal SIP algorithm (18), (12).

Similarly, if we apply the nine-diagonal SIP algorithm (11), (12) to a seven-diagonal matrix A for which (14) holds, we have the estimate (44), while if we apply the same algorithm to a seven-diagonal matrix A for which (20) holds, we have the estimate (45).

Computational results also show that the convergence of the SIP algorithm with higher accuracy in § 4 is better than that of the ordinary SIP algorithm in § 2.

Finally, we point out that the results in § 4 can also be generalized to the matrix and the SIP algorithm with more non-zero diagonals.

§ 5. Computational Results

It is interesting that although the theoretical estimates in this paper are established under the assumptions that A is strictly diagonally dominant such that (29) and (30) hold, the computational results show that the estimates still hold for the weakly diagonally dominant matrices. My colleague Mr. Liu Sing-ping has solved the Laplace equation $\Delta u=0$ in two dimensions on the unit square $0\leqslant x\leqslant 1$, $0\leqslant y\leqslant 1$ with the SIP algorithms in this paper. The results are as follows.

Table 1 ($\theta=0$)

Method	s/CPU	20	30	40	50
505	s	260	537	908	1368
	CPU	0.0628	0.281	0.831	1.96
57_{I}	s	260	537	908	1368
	CPU	0.119	0.537	1.66	3.86
57_{II}	s	107	219	368	554
	CPU	0.0504	0.227	0.672	1.63
509	s	107	219	368	554
	CPU	0.0586	0.271	0.803	1.88

Table 2 ($\theta=0.9$)

Method	s/CPU	20	30	40	50
505	s	70	137	225	336
	CPU	0.0184	0.0748	0.211	0.487
57_{I}	s	70	137	225	336
	CPU	0.0349	0.143	0.423	0.966
57_{II}	s	32	59	96	143
	CPU	0.0172	0.0662	0.184	0.436
509	s	32	59	97	144
	CPU	0.0204	0.0782	0.220	0.505

Table 3 ($\theta=0$)

Method	s/CPU	20	30	40	50
7_I7_I	s	260	537	908	1368
	CPU	0.119	0.549	1.63	3.93
7_I09	s	107	219	368	554
	CPU	0.0589	0.265	0.793	1.87
$7_{II}7_{II}$	s	107	219	368	554
	CPU	0.0497	0.223	0.667	1.65
$7_{II}09$	s	107	219	368	554
	CPU	0.0588	0.265	0.791	1.84

Table 4 ($\theta=0.9$)

Method	s/CPU	20	30	40	50
7_I7_I	s	70	137	225	336
	CPU	0.0343	0.145	0.413	0.974
7_I09	s	32	59	97	144
	CPU	0.0203	0.0778	0.220	0.503
$7_{II}7_{II}$	s	32	59	96	143
	CPU	0.0171	0.0654	0.183	0.438
$7_{II}09$	s	32	59	97	144
	CPU	0.0203	0.0778	0.220	0.504

In these tables and CPU are the numbers of iterations and the computational time (minutes) by the machine such that the iterate $u^{(s)}$ satisfies the assigned relative error, the first number 5, 7_I and 7_{II} denote that the coefficient matrix A is a five-diagonal matrix, a seven-diagonal matrix with $\zeta_{ij}\equiv 0\equiv\alpha_{ij}$ and a seven-diagonal matrix with $\xi_{ij}\equiv 0\equiv\gamma_{ij}$ respectively and the last numbers 05, 7_I, 7_{II} and 09 denote that we apply the five-diagonal, seven-diagonal 7_I, seven-diagonal 7_{II} and nine-diagonal SIP algorithms respectively to solve the problem. From these tables, we can see that the computational results and the theoretical estimates in this paper coincide very well with each other and that the SIP algorithm with $\theta\equiv 0.9$ is much better than that with $\theta\equiv 0$. On the other hand, from tables 1 and 2, we can see that if A is the classical five-diagonal matrix, it is preferable to use the 57_{II} algorithm especially in view of the number of iterations. Besides, a number of computations by Mr. Liu Sing-ping show that the SIP method is in preference to other splitting methods.

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