THE COUPLING OF BOUNDARY INTERAL AND FINITE ELEMENT METHODS FOR THE NAVIER–STOKES EQUATIONS IN AN EXTERIOR DOMAIN

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Abstract

In this paper, a technique of coupling variational formulation of FEM and BIE (boundary integral equation) is used to deal with stationary Navier-Stokes equations in an unbounded domain. We discuss well-posedness for the coupling variational problem, the regularization method and FEM-BEM approximation. Finally, operator splitting and optimal control techniques are used to treat the difficulty of nonlinearity and constraints in computer implementation.

1. Introduction

The coupling of FEM and BIE has recently been recognized as a powerful tool for solving a certain class of physical problems with an unbounded domain for which the traditional numerical analysis techniques are unsuitable.

Following basically A. Sequira etal. [1], [2] concerning Stokes case, the major aim of the present work is to develop this method for N–S equations in an unbounded domain. Essentially, the coupling method involves the choice of an artificial smooth boundary separating the unbounded domain into two regions; an integral equation over this interface, representing the solution in the exterior domain in terms of a single layer potential, is incorporated into a variational formulation in the primitive variable velocity-pressure for the interior region. This allows discretization along the artificial boundary together with a typical discretization by the FEM.

2. Statement of the Problem

The stationary N–S equations with an exterior domain are given as

$$\begin{cases} (u^j\nabla_j)u^i = \nabla_j\sigma^{ij} + f^i, \quad i = 1,2,\ldots,n, n = 2 \text{ or } 3, \quad \text{in } \Omega', \\ \text{div } u = 0 \quad \text{in } \Omega', \\ u|_\Gamma = u_0, \quad u \to u_\infty, \quad x \to +\infty, \quad \int_\Gamma u_0 ds = 0, \end{cases} \tag{2.1}$$

where Ω' is the exterior of a simply-connected bounded open set Ω in R^n with smooth boundary Γ, u the velocity of fluids, $p = p/\rho$ the pressure, f the external forces and $\lambda = \text{Re}^{-1}, \text{Re} = u_\infty L/\nu$ Reynolds number, σ^{ij}, σ_{ij} stress tensors, e^{ij}, e_{ij} strain rate tensors, ∇_i, ∇^i covariant and contravariant derivatives respectively, g_{ij}, g^{ij} metric tensors,

$$\begin{cases} \sigma_{ij}(u,p) = -pg_{ij} + 2\mu e_{ij}(u), \quad e_{ij}(u) = (\nabla_i u_j + \nabla_j u_i)/2, \\ \sigma^{ij}(u,p) = g^{ik}g^{jm}\sigma_{km}, \quad e^{ij}(u) = g^{ik}g^{jm}e_{km}. \end{cases}$$

We only consider the homogeneous boundary condition in the sequel, but all the results stated here still hold if the trace of u on Γ is any given sufficiently smooth function that admits a solenoidal extension (div $u = 0$) in Ω'.

Let $\Omega' = \Omega_1 \cup \Omega_2$ be a decomposition of the domain such that Ω_1 and Ω_2 are open subsets of Ω'. Γ_2 is their common smooth boundary with a unit normal exterior to $\Omega_2; \Omega_1$ is bounded and $\text{supp}(f) \subset\subset \Omega_1$.

It is well known [8] that there exists at least one solution for problem (2.1). Generally speaking, velosity or its gradient in subdomain Ω_2 is small in the amplitude compared with that in subdomain Ω_1. Therefore, the inertia term $u\nabla u$ in Ω_2 can be neglected, and problem (2.1) can be replaced by the following

$$\begin{cases} (u^j\nabla_j)u^i - \nabla_j\sigma^{ij}(u,p) = f^i, & \text{in } \Omega_1, \\ \text{div } u = 0, & \text{in} \Omega_1, \end{cases} \tag{2.2}$$

$$\begin{cases} \nabla_j\sigma^{ij}(u,p) = 0, & \text{in } \Omega_2, \\ \text{div } u = 0, & \text{in } \Omega_2, \end{cases} \tag{2.3}$$

$$U|_\Gamma = 0, \qquad u|^+_{\Gamma_2} = u|^-_{\Gamma_2}, \tag{2.4}$$

where the last conditions represent the appropriate assembling of the two separate problems in Ω_1 and Ω_2.

3. Variational Formulation for the Continuous Coupling Problem

In order to reduce the problem in Ω_2 into an integral equation over the boundary, the fundamental solution $\{U^{ij}, P^i\}$ of the stationary Stokes equation with the concentrated force will be employed and can be expressed in arbitrary curvilinear coordinates as [8]

$$\begin{cases} \lambda^{-1}U^{ij}(x-y) = -\frac{1}{8\pi\lambda}\{g^{ij}|x-y|^{-1} + (x^i-y^i)(x^j-y^j)|x-y|^{-3}\} \\ P^i(x-y) = \frac{1}{4\pi}\nabla^i|x-y|^{-1} \quad ,n=3 \end{cases} \tag{3.1}$$

$$\begin{cases} \lambda^{-1}U^{ij}(x-y) = -\frac{1}{4\pi\lambda}\{g^{ij}\ln|x-y|^{-1} + (x^i-y^i)(x^j-y^j)|x-y|^{-2}\} \\ P^i(x-y) = \frac{1}{2\pi}\nabla^i\ln|x-y|^{-1} \quad ,n=2 \end{cases} \tag{3.2}$$

where $|X-Y|$ is distance between the points X and Y

$$|x-y|^2 = g_{ij}(x^i-y^i)(x^j-y^j).$$

Let us denote

$$U^k(x-y) = \{U^{k_1}, U^{k_2}, U^{k_3}\}.$$

Then, Stokes problem (2.2b) can be resolved by applying the fundamental solution

$$\begin{aligned} u^i(y) = &-\int_{\Gamma_2}\sigma^{jk}(\lambda^{-1}U^i, P^i)(x-y)u_j(x)n_k(x)ds_x \\ &+\lambda^{-1}\int U^{ij}(x-y)\sigma_j(x)ds_x + c, \quad \forall y\in\Omega_2, \end{aligned} \tag{3.3}$$

$$\begin{aligned} P(y) = &-\lambda\int_{\Gamma_2} 2\nabla_i P^j(x-y)n^i(x)u_j(x)ds_x, \\ &+\int_{\Gamma_2} P^i(x-y)\sigma_i(x)ds_x, \quad \forall y\in\Omega_2, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \frac{1}{2}u^i(y) = &-\int_{\Gamma_2}\sigma^{jk}(\lambda^{-1}U^i, P^i)u_j(x)n_k(x)ds_x \\ &+\lambda^{-1}\int_{\Gamma_2} U^{ij}(x-y)\sigma_j(x)ds_x + c, \quad \forall y\in\Gamma_2 \end{aligned} \tag{3.5}$$

where c is an arbitrary constant. The kernels $U^{ij}(x-y)$ and $\sigma^{ij}(U,P)(x,y)n_j(x)$ of integral equations (3.3)–(3.5) are summable when $(x-y)\in\Gamma_2$ are close together. Indeed, $U^{ij}(x-y)$ has singularity $\ln|x-y|$(or $|x-y|^{-1}$) and we can easily see that, by elementary calculation, we have

$$\begin{cases} \begin{aligned} S^{ik}(x-y) &= \sigma^{ij}(\lambda^{-1}U^k, P^k)(x-y)n_j(x) \\ &= -\frac{1}{\pi}(x^i-y^i)(x^k-y^k)|x-y|^{-2}(x-y)n(x), \quad n=2, \\ S^{ik}(x-y) &= \sigma^{ij}(\lambda^{-1}U^k, P^k)(x-y)n_j(x) \\ &= \frac{3}{4\pi}(x^i-y^i)(x^k-y^k)|x-y|^{-5}(x-y)n(x), \quad n=3 \end{aligned} \end{cases} \tag{3.6}$$

where $(x-y)n(x) = g_{ij}(x^i-y^i)n^j(x)$ and (3.3), (3.5) are an infinitely differentiable function over the analytic boundary. Otherwise, as y approaches the boundary from the exterior, the expression (3.4) has no limit value.

Now, we introduce the following Hilbert spaces

$$X_0 = (H_0^1(\Omega_1))^n, X = (H^1(\Omega_1))^n, \quad M = L_0^2(\Omega_1) = \{q/q \in L^2(\Omega_1), \int_{\Omega_1} q dx = 0\},$$
$$W = (H_{0,\Gamma}^1(\Omega_1))^n = \{v|v \in X, v/_\Gamma = 0\}, \quad W_0 = \{v|v \in W, \text{ div } v = 0\},$$
$$T = \{\mu|\mu \in H^{-1/2}(\Gamma_2))^n, \quad \int_{\Gamma_2} \mu ds = 0\} = (\overset{\circ}{H}{}^{-1/2}(\Gamma_2))^n,$$

and bilinear and trilinear forms

$$a_0(u,v) = 2\int_{\Omega_1} e^{ij}(u)e_{ij}(v)dv = \int_{\Omega_1} \nabla u.\nabla v dv, \quad dv = \sqrt{q}dx, \tag{3.7}$$
$$a_1(u;w,v) = \int_{\Omega_1} u^j \nabla_j w^i v_i dv, \quad (p, \text{ div } v) = \int_{\Omega_1} p \text{ div } v dv, \tag{3.8}$$
$$b(\sigma,\mu) = \int_{\Gamma_2}\int_{\Gamma_2} U^{ij}(x-y)\sigma_i(x)\mu_j(y)ds_x ds_y, \quad <\sigma,\mu>_{\Gamma_2} = \int_{\Gamma_2} \sigma\mu ds, \tag{3.9}$$
$$\begin{aligned} Ku &= \left\{-\int_{\Gamma_2} \sigma^{ij}(U^m, P^m)(x-y)u_i(x)n_j(x)ds_x\right\} \\ &= \left\{-\int_{\Gamma_2} S^{im}(x-y)u_i(x)ds_x\right\}_{m=1,2,\cdots,n}, \end{aligned} \tag{3.10}$$
$$< Ku,\mu >_{\Gamma_2} = -\int_{\Gamma_2}\int_{\Gamma_2} S^{im}(x-y)u_i(y)\mu_m(x)ds_x ds_y. \tag{3.11}$$

Using (3.5) and usual methods, it is not difficult to obtain the following coupling variational formulation associated with problem (2.2) and integral equation (3.5):

$$(\mathrm{Q}) \qquad \begin{cases} \text{Find } (u,\sigma,p) \in W \times T \times M \text{ such that} \\ \lambda a_0(u,v) + a_1(u;u,v) - (p, \text{ div } v) + < v,\sigma >_{\Gamma_2} = < f,v >, & \forall v \in W, \\ \lambda^{-1}2b(\sigma,\mu) - < u,\mu >_{\Gamma_2} + 2 < Ku,\mu >_{\Gamma_2} = 0, & \forall \mu \in T, \\ \qquad (q, \text{ div } u) = 0, & \forall q \in M \end{cases}$$

or

$$(\mathrm{P}) \qquad \begin{cases} \text{Find } (u,\sigma) \in W_0 \times T \text{ such that} \\ \lambda a_0(u,v) + a_1(u;u,v) + < v,\nu >_{\Gamma_2} = < f,v >, \quad \forall v \in W_0, \\ \lambda^{-1}2b(\sigma,\mu) - < u,\mu >_{\Gamma_2} + 2 < Ku,\mu >_{\Gamma_2} = 0, \quad \forall \mu \in T. \end{cases}$$

It is well known that the bilinear form $b(.,.)$ on $T \times T$ is symmetric continuous and T-elliptic in the sense that there exists a constant $c > 0$ such that

$$b(\mu,\mu) \geq c\|\mu\|^2_{-1/2,\Gamma_2}, \quad \forall \mu \in T. \tag{3.12}$$

We emphasize that the operator defined by (3.10) is compact from W to

$(H^{1/2}(\Gamma_2))^n \subset T'$:

$$W \hookrightarrow (H^1(\Omega_1))^n \xrightarrow{\gamma_0} (H^{1/2}(\Gamma_2))^n \xrightarrow{K} (H^{3/2}(\Gamma_2))^n \overset{C}{\hookrightarrow} (H^{1/2}(\Gamma_2))^n \hookrightarrow T'$$

where $\hookrightarrow$ is the canonical injection and γ_0 is the trace operator on Γ_2. Since $H^{3/2}(\Gamma_2)$ is compactly embedded in $H^{1/2}(\Gamma_2)$, the continuity of K as well as that of the trace operator γ_0 imply the compactness of $K : W \longrightarrow T'$,

4. Existence Theorem

In order to set the coupling variational problem of the Navier-Stokes equation into an operator form, we consider the following problem: $\forall u \in W$,

$$\begin{cases} \text{Find } \sigma \in T \text{ such that} \\ b(\sigma,\mu) = \lambda/2 < u,\mu >_{\Gamma_2} - \lambda < Ku,\mu >_{\Gamma_2}, \quad \forall \mu \in T. \end{cases} \tag{4.1}$$

By virtue of (3.12), we have

Lemma 4.1. *Problem* (4.1) *admits a unique solution $\sigma \in T$, and the mapping L_0 defined by* (4.1), *$\sigma(u) = \lambda L_0(u)$, is linear continuous, and*

$$\begin{aligned} &\|\sigma(u)\|_{-1/2,\Gamma_2} \leq \lambda c(0.5 + \|K\|)\|u\|_1, \\ &\|L_0\| \leq c(0.5 + \|K\|) \ \textit{where} \ \|\cdot\| \end{aligned} \tag{4.2}$$

denotes the operator norm.

The operator L_0 possesses the following property:

Lemma 4.2. *$\forall u \in W_0$, we have*

$$< u, \sigma(u) >_{\Gamma_2} > 0 \tag{4.3}$$

where σ is defined by $\sigma^i(u) = \sigma^{ij}(u,p)n_j$, which is a surface stress tensor.

Proof. $\forall u \in W_0, \int_{\Gamma_2} u n ds = 0$. Hence we can define the Stokes problem:

$$\begin{cases} \nabla_j \sigma^{ij}(w) + \nabla^i s = 0, \ \text{div } w = 0, & \text{in } \Omega_1, \\ \qquad w|_\Gamma = 0, w|_{\Gamma_2} = u|_{\Gamma_2} & \text{on } \Gamma \cup \Gamma_2 \end{cases}$$

which has a unique solution. In virtue of the symmetry of tensor σ^{ij} and

$$g^{ij}e_{ij}(w) = g^{ij}\nabla_i w_j = \text{ div } w = 0, \quad (\nabla^i s, w_i) = 0,$$

we have

$$\begin{aligned} < u, \sigma(u) >_{\Gamma_2} &= \int_{\Gamma_2} \sigma^{ij}(u,p) u_i n_j ds = \int_{\Gamma_2} \sigma^{ij}(w,s) w_i n_j ds = \int_{\Omega_1} \nabla_j(\sigma^{ij} w_i) dv \\ &= (\nabla^i s, w_i) + \int_{\Omega_1} \sigma^{ij} e_{ij}(w) dv = \int_{\Omega_1} (-s g^{ij} \nabla_i w_j + \lambda e^{ij} e_{ij}) dv \\ &= \lambda \int_{\Omega_1} e_{ij} e^{ij} dv > 0. \end{aligned}$$

The proof is completed.

Now we define a bilinear form on $W \times W$:

$$A_0(\xi, v) = a_0(\xi, v) + \frac{1}{\lambda} < (v, \sigma(\xi) >_{\Gamma_2} = a_0(\xi, v) + < v, L_0(\xi) >_{\Gamma_2}, \forall \xi, v \in W_0. \quad (4.4)$$

In view of lemma 4.2, the form $A_0(\cdot, \cdot)$ is continuous linear and W_0–elliptic:

$$A_0(\xi, \xi) \geq \lambda \|\xi\|^2_{1,\Omega_1}, \quad \forall \xi \in W_0. \quad (4.5)$$

Hence, the following variational problem : $\forall F \in W'$,

$$\begin{cases} \text{Find } \xi \in W_0 \text{ such that} \\ \lambda A_0(\xi, v) = < F, v >, \forall v \in W_0 \end{cases} \quad (4.6)$$

has a unique soluton. Therefore, $F \in W' \to \xi = LF$ can be defined by (4.6).

On the other hand, by the trilinearity of $a_1(.;.,.)$, there exists $G(u) \in w'$ for $u \in W$ such that

$$< G(u), v > = a_1(u; u, v) \quad , \forall v \in W \quad (4.7)$$

and $G(u) \in (L^{4/3}(\Omega_1))^n$ (cf. Li Kaitai [4]),

$$\|G(u)\|_* \leq N|u|^2_{1,\Omega_1}, \qquad N = \|a_1\|.$$

Obviously, $w \in W_0, < f, v > -a_1(w; w, v)$ is a linear continuous functionals. So the following variational problem : $\forall w \in W_0$,

$$\begin{cases} \text{Find } \xi \in W_0 \text{ such that} \\ \lambda A_0(\xi, v) = < f, v > -a_1(w; w, v), \quad \forall v \in W_0 \end{cases} \quad (4.8)$$

has a unique solution $\xi(w) = L(f - G'(w)) = Lf - LG(w)$, and

$$|\xi(w)|_{1,\Omega_1} \leq \lambda^{-1}\{\|f\|_* + N|w|^2_{1,\Omega_1}\}.$$

Set

$$Tw = LG(w), \quad \forall w \in W. \quad (4.9)$$

Then, problem (4.8) can be expressed in an operator form

$$\xi(w) = \tilde{f} + Tw, \quad \tilde{f} = Lf. \quad (4.10)$$

It is clear that (4.6) is a coupling variational problem for Stokes equations. From the regularity result of coupling for Stokes problems (cf. A. sequeira [1]), we have $\xi \in (H^{2,4/3}(\Omega_1))^n \cap W_0$. Since $(H^{2,4/3}(\Omega_1))^n$ and $(H^2(\Omega_1))^n$ are compactly embedded into $(H^1(\Omega_1))^n$, we can conclude that the mapping $w \to \xi(w)$ is compact.

Furthermore, it is easy to prove that $\xi(.)$ is Lipschitz continuous:

$$|\xi(w_1)-\xi(w_2)|_{1,\Omega_1} \le \lambda^{-1}N\{|w_1|_{1,\Omega_1}+|w_2|_{1,\Omega_1}\}|w_1-w_2|_{1,\Omega_1}, \quad \forall w_1, w_2 \in W_0. \tag{4.13}$$

In addition, if f satisfies the condition

$$4N\lambda^{-2}\|f\|_* < 1, \tag{4.14}$$

then $\xi(.)$ is a contraction mapping from a closed set S into S. Therefore, we have

Lemma 4.3. *The mapping $w \to \xi(w)$ defined by (4.10) is locally Lipschitz continuous and compact from W_0 into W_0 if $f \in (L^2(\Omega_1))^n$.*

Furthermore, if f satisfies condition (4.14), then the variational problem

$$\begin{cases} \text{Find } u \in W_0 \text{ such that} \\ \lambda A_0(u,v) + a_1(u;u,v) = < f, v >, \forall v \in W_0 \end{cases} \tag{4.15}$$

has a unique solution and

$$|u|_{1,\Omega_1} \le \frac{1}{2\lambda}\|f\|_*. \tag{4.16}$$

From Lemmas 4.3 and 4.1, we conclude that problem (P) has a unique solution if condition (4.14) is satisfied. Owing to the equivalence between problems (P) and (Q), we have

Theorem 1. *Assume $f \in W'$ and (4.14) holds. Then the variational problem (Q) has exactly one solution $(u,\sigma,p) \in W_0 \times T \times M$. Moreover, if $f \in (H^{m-1}(\Omega_1))^n$, then $u \in (H^{m+1}(\Omega_1))^n, \sigma \in (H^{m-1/2}(\Gamma_2))^n, p \in H^m(\Omega_1)$ and there exists a constant $c > 0$ such that*

$$\|u\|_{m+1,\Omega_1} + \|\sigma\|_{m-1/2,\Gamma_2} + \|p\|_{m,\Omega_1} \le c\|f\|_{m-1,\Omega_1}.$$

Proof. The proof is rather technical and can be omitted.

5. Approximation of Branches of Nonsingular Solutions

5.1. Regularization Method

Let us consider an abstract nonlinear problem and its approximation. Assume X, Y, Z are three Banach spaces and there exists a compact interval of the line R. Given a c^p-mapping $(p \ge 1)$

$$F: (\lambda, u) \in \Lambda \times X \to F(\lambda, u) \in Y$$

we want to solve the equation

$$F(\lambda, u) = u + LG(\lambda, u) = 0 \tag{5.1}$$

where $L \in \mathcal{L}(Z;Y)$, and G is a c^2-mapping from $\Lambda \times X$ into Z. We are interested in the branch $u(\lambda)$ of nonsingular solutions of (5.1), *i.e.*

$$D_uF(\lambda,u(\lambda)) \text{ is an isomorphism from } X \text{ onto } Y \text{ for all } \lambda \in \wedge. \tag{5.2}$$

We will make an additional assumption:

$$\left\{\begin{array}{l}\text{There exists another Banach space } V \text{ contained in } Z \text{ with continuous}\\ \text{imbedding such that } D_uG(\lambda,u) \in \mathcal{L}(X;V);\ \ \forall\lambda \in \Lambda,\ \ \forall u \in X.\end{array}\right. \tag{5.2}$$

To approximate (5.1), we introduce an operator $L_h \in \mathcal{L}(Z,X)$ to approximate the problem:

$$F_h(\lambda,u) = u + L_hG(\lambda,u). \tag{5.4}$$

The approximation problem is

$$\text{Find } u_h \in X \text{ such that } F_h(\lambda,u_h) = 0. \tag{5.5}$$

Furthermore, we assume that operator L_h has the following properties:

$$\lim_{h\to 0} \|(L-L_h)g\|_Z = 0, \quad \forall g \in X, \tag{5.6}$$

$$\lim_{h\to 0} \|(L_h-L)\|_{\mathcal{L}(V;X)} = 0. \tag{5.7}$$

If the imbedding of V into Z is compact, then (5.7) is a consequence of (5.6).

The following theorem (Girault and Raviart [3]) will be used:

Theorem 2. *Assume that G is c^2-mapping from $\Lambda \times X$ into Z and the mapping D_uG is bounded on all bounded subsets of $\Lambda \times X$. Assume in addition that conditions (5.3), (5.6) and (5.7) hold and that $\{(\lambda,u(\lambda));\lambda \in \Lambda\}$ is a branch of nonsingular solutions of (5.1). Then there exists a neigborhood ϑ of the origin in X and, for $h \le h_0$ small enough, a unique c^2-function $\lambda \in \Lambda \to u_h(\lambda) \in X$ such that*

$$\{(\lambda,u_h(\lambda));\lambda \in \Lambda\} \textit{ is a branch of nonsingular solutions of } (5.4)\ , \tag{5.8}$$

$$u_h(\lambda) - u(\lambda) \in \vartheta \textit{ for all } \lambda \in \Lambda. \tag{5.9}$$

Furthermore, there exists a constant $c > 0$ independent of h and λ with

$$\|u_h(\lambda) - u(\lambda)\|_X \le c\|(L-L_h)G(\lambda,u(\lambda))\|_X, \quad \forall\lambda \in \Lambda. \tag{5.10}$$

If G is a c^p-mapping (with $p \ge 2$) and D_uG is bounded on all bounded subsets of $\Lambda \times X$, then $u(\lambda) \in c^p(\Lambda;X)$ and

$$\left\|\frac{d^m}{d\lambda^m}(u(\lambda)-u_h(\lambda))\right\|_X \le c_m\sum_{l=0}^{m}\left\|(L-L_h)\frac{d^l}{d\lambda^l}G(\lambda;u(\lambda))\right\|_X, \quad 0 \le m \le p-1.$$

Now we consider the penalty method for coupling Stokes problems. Assume U and $U_* \in X$ are respectively a solution of coupling Stokes problem

$$\begin{cases} \text{Find } U = (u, \sigma, p) \in X \text{ such that} & \\ \lambda A_0(u, v) - (p, \mathrm{div} v) = < f, v >, & \forall v \in W, \\ \lambda^{-1} b(\sigma, \mu) - 0.5 < u, \mu > + < Ku, \mu >_{\Gamma_2} = 0 & \forall \mu \in T, \\ \qquad (q, \mathrm{div} u) = 0, & \forall q \in M, \end{cases} \tag{5.11}$$

and a solution of regularized coupling Stokes problem