ON BOUNDARY INTEGRAL EQUATIONS OF THE FIRST KIND [1)]

GEORGE C. HSIAO
(Department of Mathematical Sciences, University of Delaware Newark, DE 19716, USA)

Abstract

A large class of elliptic boundary value problems in elasticity and fluid mechanics can be reduced to systems of boundary integral equations of the first kind. This paper summarises some of the basic concepts and results concerning the mathematical foundation of boundary element methods for treating such a class of boundary integral equations.

§1. Introduction

The boundary integral equation method for numerical solutions to elliptic boundary value problems has received much attention and gained wide acceptance in recent years. As is well known, the method is particularly suitable for obtaining numerical solutions of exterior boundary value problems and implies an approximate technique by which the problem dimensions are reduced by one. The latter leads to an appreciable reduction in the numbers of algebraic equations generated for solutions, as well as much simplified data presentation. However, irrespective of the particular numerical implementation chosen, central to the method is the reduction of boundary value problem to equivalent boundary integral equations over the boundary of the domain for the problems under consideration. This reduction is by no means unique.

In the conventional approach, Fredholm integral equations of the second kind are generally obtained either by using the "direct method" based on Green's formula or the "indirect method" in which case solutions are expressed in terms of simple or double layer potentials depending on the problem under consideration. The integral equations of the second kind are numerically stable and hence have been used extensively in engineering applications. However, in contrast to the partial differential

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equations, all the essential properties of the original elliptic operators such as symmetry and coerciveness are generally not preserved for the corresponding boundary integral operators in the variational formulation. Hence from the theoretical and computational points of view, the boundary element method for the Fredholm integral equations of the second kind is not satisfactory.

Alternatively, for a variety of physical problems with the Dirichlet data, if one expresses the solutions in terms of simple-layer potentials (Fichera[3], Fichera and Ricci[4], Hsiao[10–11], Hsiao and MacCamy [21–22], Hsiao and Wendland[25–30], LeRoux[31], and MacCamy[32]) or employs Green's formula for the solutions (Hsiao and Roach[24], Nedelec and Planchard[33]), boundary integral equations of the first kind will result. Similarly for problems with the Neumann data, boundary integral equations of the first kind (involving hypersingular integral operators) can be obtained by using double–layer potentials or by differentiating the Green representation formula for the solutions (Feng[6–7], Giroire and Nedelec[8], Han[9], Hsiao[16], Hsiao and Wendland[27, 29–30], and Wendland[36]). In these formulations, in contrast to the integral equations of the second kind, the symmetry and coerciveness properties of the integral operators follow directly from those of the original partial differential operators via the trace theorem in Sobolev spaces and vice versa. Hence the boundary element method for the integral equations of the first kind is more satisfactory and compatible with the finite element method for the partial differential equations.

In this paper, for simplicity we will confine to the model problems for the Laplacian in $\mathbb{R}^n, n = 2, 3$, the exterior Dirichlet and Neumann Problems. In either case we will reduce the boundary-value problem to a boundary integral equation of the first kind via the direct or indirect method. As will be seen, the corresponding boundary integral operators are typical pseudodifferential operators of order 2α where α is equal to $-\frac{1}{2}$ for the Dirichlet problem and $+\frac{1}{2}$ for the Neumann problem.

§2. Boundary Integral Equations

Throughout the paper, let Γ be a sufficiently smooth simple closed curve in $\mathbb{R}^2$ or surface in $\mathbb{R}^3$, and let Ω^c be the exterior domain. We consider two fundamental boundary value problems for the Laplace equation

$$\Delta u = 0 \quad \text{in} \quad \Omega^c, \tag{1}$$

the *Dirichlet problem* (D) and the *Neumann problem* (N). The boundary conditions and the conditions at infinity are

$$u|_\Gamma = \phi \quad \text{on} \quad \Gamma \quad \text{and} \quad u = O(|x|^{2-n}) \quad \text{as} \quad |x| \to \infty \tag{2}$$

for (D), and

$$\frac{\partial}{\partial\nu}u|_\Gamma = \psi \quad \text{on} \quad \Gamma \quad \text{and} \quad u = O(|x|^{-1}) \quad \text{as} \quad |x| \to \infty \tag{3}$$

for (N). Here ν denotes the exterior normal to Γ with respect to $I\!R^n \setminus \overline{\Omega}^c, n = 2, 3$, and the given data ϕ and ψ will satisfy certain regularity conditions to be specified later. For (N) in $I\!R^2$, we also assume that ψ satisfies the compatibility condition:

$$\int_\Gamma \psi ds = 0, \tag{4}$$

which is necessary for the existence of a unique solution to (1) and (3).

To reduce the boundary value problems to boundary integral equations of the first kind, we begin with the Green representation formula for the unknown solution $u(x)$ of (1):

$$u(x) = \int_\Gamma \Big\{\frac{\partial}{\partial\nu_y}\gamma_n(x,y)\sigma_1(y) - \gamma_n(x,y)\sigma_2(y)\Big\} ds_y - \omega, \quad x \in \Omega^c, \tag{5}$$

where

$$\gamma_2(x,y) = -\frac{1}{2\pi}\log|x-y| \text{ and } \gamma_3(x,y) = \frac{1}{4\pi}|x-y|^{-1}$$

are the fundamental solutions of the Laplacian in $I\!R^2$ and $I\!R^3$. Here $\sigma_1 = u|_\Gamma$ and $\sigma_2 = \frac{\partial}{\partial\nu}u\Big|_\Gamma$ are the Cauchy data of the solution of (1), and ω is an unknown constant which is zero except for (D) in $I\!R^2$. The Cauchy data σ_1 and σ_2 are not linearly independent and they are related by the boundary integral equatións derived from (5) by taking the traces of u on Γ:

$$\frac{1}{2}\sigma_1 = \mathbf{K}_n\sigma_1 - \mathbf{V}_n\sigma_2 - \omega \quad \text{and} \quad \frac{1}{2}\sigma_2 = -\mathbf{W}_n\sigma_1 - \mathbf{K}'_n\sigma_2 \tag{6}$$

or simply $\boldsymbol{\sigma} = \mathbf{C}_n\boldsymbol{\sigma} - \boldsymbol{\omega}$ if we denote $\boldsymbol{\sigma} = (\sigma_1,\sigma_2)^T$, $\boldsymbol{\omega} = 2(\omega,0)^T$ and $\mathbf{C}_n$ the Calderón projector[2–3]. Here $\mathbf{K}_n, \mathbf{V}_n, \mathbf{W}_n$ and $\mathbf{K}'_n$ are, respectively the boundary integral operators of the *double, simple, hypersingular* and *adjoint of the double layer potentials:* For $x \in \Gamma$, they are defined by

$$\begin{aligned}
\mathbf{K}_n\sigma(x) &:= \int_\Gamma \frac{\partial}{\partial\nu_y}\gamma_n(x,y)\sigma(y)ds_y;\\
\mathbf{V}_n\sigma(x) &:= \int_\Gamma \gamma_n(x,y)\sigma(y)ds_y,\\
\mathbf{W}_n\sigma(x) &:= -\frac{\partial}{\partial\nu_x}\int_\Gamma \frac{\partial}{\partial\nu_y}\gamma_n(x,y)\sigma(y)ds_y;\\
\mathbf{K}'_n\sigma(x) &:= \int_\Gamma \frac{\partial}{\partial\nu_x}\gamma_n(x,y)\sigma(y)ds_y.
\end{aligned} \tag{7}$$

These are four basic boundary integral operators whose properties will be discussed later.

For the *Dirichlet problem* (D), since $\sigma_1 = \phi$ is given, from the first equation of (6), we arrive at the boundary integral equation for σ_2 and for the unknown constant $\omega(\omega = 0$, if $n \neq 2)$:

$$\mathbf{V}_n\sigma_2 + \omega = \tilde{\phi} := -\frac{1}{2}\phi + \mathbf{K}_n\phi. \tag{8}$$

For $n = 2$, the nuknown σ_2 is also required to satisfy the normalization condition

$$\int_\Gamma \sigma_2 ds = 0, \tag{9}$$

because of the growth condition in (3). This formulation based on (5), (8) with $\omega = 0$ for $n = 3$ or (5), (8), (9) for $n = 2$ is the *direct approach*. For the *indirect approach*, we may seek a solution in the form of the simple-layer potential (5) with $\sigma_1 = 0$, and arrive at the similar equation (8) or equations (8) and (9) if we replace $\tilde{\phi}$ by $-\phi$ in (8).

On the other hand, for the *Neumann problem* (N), $\sigma_2 = \psi$ is given and we have the boundary integro-differential equations for σ_1:

$$\mathbf{W}_n\sigma_1 = \tilde{\psi} := -\frac{1}{2}\psi - \mathbf{K}'_n\psi, \tag{10}$$

$$\int_\Gamma \sigma_1 ds = b \tag{11}$$

in the *direct approach* for both $n = 2$ and $n = 3$, based on the representation (5) with $w = 0$. In this formulation, we have added the normalization condition (11) in order to insure the uniqueness of the solution. (Note that nonzero constants are eigensolutions of (10)) Here b is any fixed constant. Similarly, for $n = 2$, in the *indirect approach,* we may use the representation formula (5) with $\sigma_2 = 0, w = 0$, and obtain the same boundary integral equations (10) and (11) except that $\tilde{\psi}$ is now replaced by $-\psi$. However, for $n = 3$, because of the growth condition (3), we seek a solution in the form:

$$u(x) = -\int_\Gamma \frac{\partial}{\partial u_y}\gamma_3(x,y)\sigma_1(y)ds_y + \alpha|x|^{-1}, \quad x \in \Omega^c \tag{12}$$

with a constant α to be determined. Then the boundary condition (3) yields the same equation (10) for the unknown density function σ_1 but with $\tilde{\psi}$ replaced by

$$\tilde{\psi} := \psi - \alpha\frac{\partial}{\partial v_x}|x|^{-1}, \quad x \in \Gamma. \tag{13}$$

Now, the constant α is chosen so that

$$\int_\Gamma \tilde{\psi} ds = 0 \quad \text{or} \quad \alpha = -\frac{1}{4\pi}\int_\Gamma \psi ds.$$

We note that in $\mathbb{R}^3$ the compatibility condition (4) is not needed. However, for the uniqueness, we impose here again the condition (11).

Equations (8), (9) (or simply (8) for $n = 3$) and (10) (11) may serve as representative boundary integral equations of the first kind for the elliptic boundary value problems in $\mathbb{R}^n, n = 2$ and 3. If we introduce the operators $\mathbf{A}_D, \mathring{\mathbf{A}}_D$ and $\mathbf{A}_N$:

$$\mathbf{A}_D(\sigma, w) := (\mathbf{V}_2\sigma + w \int_\Gamma \sigma ds), \quad \mathring{\mathbf{A}}_D\sigma := \mathbf{V}_3\sigma \tag{14}$$

$$\mathbf{A}_N(\sigma, w) := (\mathbf{W}_n\sigma + w, \int_\Gamma \sigma ds), \quad n = 2, 3,$$

then these equations may be rewritten in the form

$$\mathbf{A}_D(\sigma, w) = (\tilde{\phi}, 0)(n = 2), \mathring{\mathbf{A}}_D\sigma = \tilde{\phi}(n = 3) \tag{15}$$

and

$$\mathbf{A}_N(\sigma, w) = (\tilde{\psi}, b), \quad n = 2, 3 \tag{16}$$

for (D) and (N), respectively.

We remark that (16) is indeed equivalent to the equations (10) (11) for the *Neumann problem* (N), since under the assumptions (4) and (13), one can easily show that $\omega = 0$. However, in the present formulation not only the test function space is relaxed but also, as will be seen, the algebraic system in the Galerkin approximation below will not become overdetermined by including this fictitious constant ω (see Hsiao[13]).

§3. Mapping Properties

We now consider the existence and uniqueness of the solutions of equations (15) and (16). For this purpose, we need to examine the mapping properties of the integral operators defined by (7). First, we need some notation. For $s \geq 0$, we denote by $H^s(\Gamma)$, the Sobolev spaces (or their interpolation spaces) of generalized functions on Γ, and for $s < 0$, let $H^s(\Gamma)$ be the dual of $H^{-s}(\Gamma)$. The norm in $H^s(\Gamma)$ will be denoted by $\|\cdot\|_s$. Thus, in the case $s = 0, \|\cdot\|_0$ is the L_2-norm, and $(\cdot,\cdot)_0$ is the corresponding inner product.

The following mapping properties are now well known:

Theorem 1. *For C^∞ boundary Γ, the boundary integral operators defined by* (7)

$$\mathbf{V}_n : H^{s-1/2}(\Gamma) \to H^{s+1/2}(\Gamma); \quad \mathbf{K}_n : H^{s+1/2}(\Gamma) \to H^{s+3/2}(\Gamma)$$
$$\mathbf{K}'_n : H^{s-1/2}(\Gamma) \to H^{s+1/2}(\Gamma); \quad \mathbf{W}_n : H^{s+1/2}(\Gamma) \to H^{s-1/2}(\Gamma)$$

are continuous for any $s \in \mathbb{R}$.

The mapping properties are particularly important for $s = 0$, in which case we will find the traces of the variational solutions of the boundary value problems from the integral representation formulas. In terms of the terminology in pseudodifferential operators[35]. $\mathbf{V}_n, \mathbf{K}_n$ and $\mathbf{K}'_n$ are operators of order -1, while $\mathbf{W}_n$ is of order $+1$. In fact, both $\mathbf{V}_n$ and $\mathbf{W}_n$ are strongly elliptic pseudodifferential operators on boundary manifold Γ (see e.g., [3], [27] and [37]), and they are related according to

$$(\mathbf{W}_2\sigma, \phi)_0 = (\mathbf{V}_2\sigma', \phi')_0 \quad \text{and} \quad (\mathbf{W}_3\sigma, \phi)_0 = (\mathbf{V}_3\nu \wedge \nabla\sigma, \nu \wedge \nabla\phi)_0$$

for all $\sigma, \phi \in H^{1/2}(\Gamma)$. Here the prime denotes the tangential derivative with respect to the arc length.

For the systems (15) and (16), we have the following existence and uniqueness results. The proofs may be found in Hsiao and Wendland[25], Hsiao[14, 16], and Wendland[36].

Theorem 2. (a) *The integral operator $\mathring{\mathbf{A}}_D$ is continuous and bijective from $H^{s-1/2}(\Gamma)$ onto $H^{s+1/2}(\Gamma)$ for any $s \in \mathbb{R}$ and it is $H^{-1/2}(\Gamma)$-elliptic, i.e. there exists a constant $\gamma > 0$ independent of σ such that the strong coerciveness property:*

$$(\mathring{\mathbf{A}}_D\sigma, \sigma)_0 \geq \gamma\|\sigma\|^2_{-1/2} \tag{17}$$

holds $\forall\sigma \in H^{-1/2}(\Gamma)$. (b) *The operators $\mathbf{A}_D : H^{s-1/2}(\Gamma) \times \mathbb{R} \to H^{s+1/2}(\Gamma) \times \mathbb{R}$ and $\mathbf{A}_N : H^{s+1/2}(\Gamma) \times \mathbb{R} \to H^{s-1/2}(\Gamma) \times \mathbb{R}$ are continuous and bijective for any $s \in \mathbb{R}$. Moreover, the following Gårding's inequalities hold:*

$$(\mathbf{A}_D(\sigma, \omega), (\sigma, \omega))_0 \geq \gamma\{\|\sigma\|^2_{-1/2} + |\omega|^2\} - \delta\{\|\sigma\|^2_{-1} + |\omega|^2\} \tag{18}$$

$\forall(\sigma, \omega) \in H^{-1/2} \times \mathbb{R}$, and

$$(\mathbf{A}_N(\sigma, \omega), (\sigma, \omega))_0 \geq \gamma\{\|\sigma\|^2_{1/2} + |\omega|^2\} - \delta\{\|\sigma\|^2_0 + |\omega|^2\} \tag{19}$$

$\forall(\sigma, \omega) \in H^{1/2}(\Gamma) \times \mathbb{R}$, where $\gamma > 0$ and $\delta \geq 0$ are constants independent of σ and ω.

Theorem 2 implies that the operators here are isomorphisms (hence the inverses of the operators exist and are bounded) between the corresponding function spaces. From the inequalities (17)–(19), we see that the operators $\mathring{\mathbf{A}}_D, \mathbf{A}_D$ and $\mathbf{A}_N$ are Fredholm operators of index zero for $S = 0$. Because of the strong ellipticity, they are, in fact, Fredholm operators of index zero for all $s \in \mathbb{R}$ (see Treves [35, Thm.2.5]). Hence, the classical Fredholm alternative remains valid here. The function spaces $H^{-1/2}(\Gamma)$, $H^{-1/2}(\Gamma) \times \mathbb{R}$ and $H^{1/2}(\Gamma) \times \mathbb{R}$ are the corresponding energy spaces. We remark that here $(\cdot, \cdot)_0$ actually represents the natural L_2-duality pairing between the energy space and its dual. Furthermore, for $n = 3$, if we denote the simple-layer potential by

$$v(x) = \int_\Gamma \gamma_3(x, y)\sigma(y)ds, \quad x \in \mathbb{R}^3 \backslash \Gamma$$

and denote the jump of $\frac{\partial}{\partial\nu}v$ across Γ by $\left[\frac{\partial}{\partial\nu}\right]$, then one can show that

$$(\mathbf{V}_3\sigma,\sigma)_0=\int_\Gamma v\left[\frac{\partial v}{\partial\nu}\right]ds=\int_{\mathbb{R}^3}|\nabla v|^2dx\cong\|v\|^2_{H^1_\rho(\mathbb{R}^3)},$$

where $\|v\|^2_{H^1_\rho(\mathbb{R}^3)}:=\|\rho v\|^2_{L^2(\mathbb{R}^3)}+\|\nabla v\|^2_{L^2(\mathbb{R}^3)}$, and $\rho=(1+|x|^2)^{-1/2}$ is the weight function. The coerciveness property (17) then follows easily from the trace theorem and boundedness of the inverse of $\mathbf{V}_3$. However, for $n=2$, we must restrict σ to some subspace of $H^{-1/2}(\Gamma)$ in order to employ the same approach[3], since the corresponding Dirichlet integral generally does not exist for $\sigma\in H^{-1/2}(\Gamma)$ and $\mathbf{V}_2$ is not one-to-one in general. Nevertheless, we may also have the strong coerciveness property for $\mathbf{V}_2$, provided diam $(\mathbb{R}^2\backslash\overline{\Omega}^c)<1$ (see Hsiao and Wendland[25]).

§4. Galerkin Approximations

In what follows, we rewrite (15) and (16) in a more general form

$$\mathbf{A}(\sigma,\omega)=(f,b) \tag{20}$$

with given (f,b) and consider $\mathbf{A}:H^s(\Gamma)\times\mathbb{R}\to H^{s-2\alpha}(\Gamma)\times\mathbb{R}, s\in\mathbb{R}$ as a continuous, bijective map satisfying the Gårding inequality of the form

$$(\mathbf{A}(\sigma,\omega),(\sigma,\omega))_0\ge\gamma\{\|\sigma\|^2_\alpha+|\omega|^2\}-|\mathbf{K}[(\sigma,\omega),(\sigma,\omega)]| \tag{21}$$

for all $(\sigma,\omega)\in H^\alpha(\Gamma)\times\mathbb{R}$, where $\mathbf{K}$ is a compact bilinear form on $H^\alpha(\Gamma)\times\mathbb{R}$. The operator $\mathbf{A}$ is of order 2α , and $H^\alpha(\Gamma)\times\mathbb{R}$ is the energy space. In particular, for $\mathbf{A}=\mathbf{A}_D$ and $\mathbf{A}=\mathbf{A}_N$, we have $2\alpha=-1$ and $+1$; respectively. Note that the imbeddings $H^{-1/2}(\Gamma)\times\mathbb{R}\hookrightarrow H^{-1}(\Gamma)\times\mathbb{R}$ and $H^{1/2}(\Gamma)\times\mathbb{R}\hookrightarrow H^0(\Gamma)\times\mathbb{R}$ are both compact. Hence the formulation (20), indeed, includes (15) and (16) as special examples.

The Gårding inequality implies that $\mathbf{A}$ is a Fredholm operator of index zero, and it plays a fundamental role for asymptotic error estimates for boundary element methods. To describe the Galerkin method, we need the *variational formulation* of (20). *Given* $(f,b)\in H^{s-2\alpha}(\Gamma)\times\mathbb{R}$, *we say that* (σ,ω) *is a variational (or weak) solution of (20) if* $(\sigma,\omega)\in H^s(\Gamma)\times\mathbb{R}$ *and satisfies*

$$a((\sigma,\omega),(\chi,\kappa))=\ell(\chi,\kappa)\quad\forall\quad(\chi,\kappa)\in H^s(\Gamma)\times\mathbb{R}, \tag{22}$$

where the bilinear form $a(\cdot,\cdot)$ *and the continuous linear functional* $\ell(\cdot)$ *are defined by*

$$a((\sigma,\omega),(\chi,\kappa)):=(\mathbf{A}(\sigma,\omega),(\chi,\kappa))_0$$

and

$$\ell(\chi,\kappa) := ((f,b),(\chi,\kappa))_0 = (f,\chi)_0 + b\cdot\kappa.$$

We remark that in our formulation the pairs $H^s(\Gamma)$ and $H^{s-2\alpha}(\Gamma)$ are dual spaces with respect to the L_2-scalar product only for $s=\alpha$. This can be easily modified by using the $(\cdot,\cdot)_{s-\alpha}$ scalar product. However, we will not adopt this approach here[36].

Galerkin's method consists of seeking an approximate solution to (22) in a finite dimensional subspace $S_h \times I\!R$ of the space $H^s(\Gamma)\times I\!R$ of admissible functions rather than in the whole space. More preccisely, for fixed $s\geq\alpha$, let $S_h\subset H^s(\Gamma)$ be a family of finite dimensional subspaces which approximate $H^\alpha(\Gamma)$, i.e. for every $\sigma\in H^\alpha(\Gamma)$ there exists a sequence $\chi\in S_h\subset H^s(\Gamma)\subset H^\alpha(\Gamma)$ such that $\|\chi-\sigma\|_\alpha\to 0$ as $h\to 0$. Here h denotes a parameter which is inversely proportional to the dimension of S_h. Then the *Galerkin approximation of the solution* (σ,ω) *of* (22) *is the solution pair* $(\tilde\sigma,\tilde\omega)\in S_h\times I\!R$ *satisfying the Galerkin equation*

$$a((\tilde\sigma,\tilde\omega),(\chi,\kappa)) = \ell(\chi,\kappa)\quad \forall(\chi,\kappa)\in S_h\times I\!R. \tag{23}$$

The essential properties concerning the Galerkin approximation $(\tilde\sigma,\tilde\omega)$ can be summarized in the following theorem without specifying the subspace S_h in any particular form (Hisao and Wendland[25], and Stephan and Wendland[34]).

Theorem 3. *To* $\mathbf{A}$ *in* (20), *there exists an* $h_0>0$ *such that the corresponding Galerkin equations* (23) *admit a unique solution pair* $(\tilde\sigma,\tilde\omega)$ *for every* $h\leq h_0$. *Moreover, the Galerkin projections*

$$\mathbf{G}_\alpha : (\sigma,\omega)\to(\tilde\sigma,\tilde\omega)$$

is uniformly bounded, that is,

$$\|\mathbf{G}_\alpha\|_{\mathbf{H}_E,\mathbf{H}_E} := \sup_{\|\sigma\|_\alpha+|\omega|\leq 1}\|\mathbf{G}_\alpha(\sigma,\omega)\|_{\mathbf{H}_E}\leq c$$

for all $h\leq h_0$, *where* $c=c(h_0)$ *and* $\mathbf{H}_E := H^\alpha(\Gamma)\times I\!R$. *Consequently, we have Céa's type of inequality:*

$$\|\sigma-\tilde\sigma\|_\alpha + |\omega-\tilde\omega| \leq (1+c)(\|\sigma-\chi\|_\alpha + |\omega-\kappa|) \tag{24}$$

for all $(\chi,\kappa)\in S_h\times I\!R$.

The uniform boundedness of the Galerkin projections follows, of course, mainly from the Gårding inequality (21). The significance of (24) is that it provides the basic inequality for obtaining convergence results in norms other than the energy norm for the integral operator $\mathbf{A}$. As in the case of partial differential equations, this simple, yet crucial estimate, (24), shows that the problem of estimating the error between the solution (σ,ω) and the Galerkin approximation $(\tilde\sigma,\tilde\omega)$ is reduced to a problem in the approximation theory. In particular, if we assume that the finite-dimensional subspace S_h is a regular finite element space in the sense of Babuška and Aziz[1], then one may obtain convergence results with respect to a whole range of Sobolev space norms, including super-convergence results by using Aubin-Nitshe Lemma for the boundary integral equations. Details may be found in the references[24].

Stability analysis for boundary integral equations of the first kind (20) is also available in the references[12, 14, 23, 26]. It is now well known that Fredholm integral equations of the first kind are generally ill-posed in the sense that solutions do not depend continuously on the given data in appropriate function spaces, if the corresponding integral operators have negative orders such as (20) with negative α. In this case $\mathbf{A} : H^s(\Gamma) \times I\!R \to H^s(\Gamma) \times I\!R$ is *compact* because of the compact imbedding $H^{s-2\alpha}(\Gamma) \hookrightarrow H^s(\Gamma)$ for $\alpha < 0$. Consequently $\mathbf{A}^{-1}$ is not bounded from $H^s(\Gamma) \times I\!R$ into itself. In turn, this may cause problems of instability when one performs the numerical computations. Nevertheless, it is shown in Hsiao[14] that an optimal choice of the mesh size can be made in the numerical computations so that one will obtain an optimal rate of convergence of the approximate solutions. On the other hand, for $\alpha > 0, \mathbf{A}^{-1}$ is *compact* but $\mathbf{A}$, like differential operators, is unbounded from $H^s(\Gamma) \times I\!R$ into itself. Thus, the L_2-condition number of the discrete equation (23) is always of $O(h^{-|2\alpha|})$ in spite of the sign of α (see Hsiao[16] for details).

To conclude the paper, we comment that the approach here based on the formulation of boundary integral equations of the first kind has been employed in a variety of problems in elasticity and fluid mechanics including numerical experiments. In this connection, we refer the reader to the references[15, 17–22, 28–30].

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